

AMP ALGORITHMS AND STEIN'S METHOD: UNDERSTANDING TAP EQUATIONS WITH A NEW METHOD

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ABSTRACT. We propose a new iterative construction of solutions of the classical TAP equations for the Sherrington-Kirkpatrick model, i.e. with finite-size Onsager correction. The algorithm can be started in an arbitrary point, and converges up to the AT line. The analysis relies on a novel treatment of mean field algorithms through Stein's method. As such, the approach also yields weak convergence of the effective fields at all temperatures towards Gaussians, and can be applied, upon proper alterations, to all models where TAP-like equations and a Stein-operator are available.

1. INTRODUCTION AND MAIN RESULTS

Let $N \in \mathbb{N}$ and consider independent standard Gaussians $\mathbf{G} \equiv \{g_{ij}\}_{1 \leq i < j \leq N}$ issued on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We set $g_{ij} = g_{ji}$ for $i > j$. For simplicity of notation, we set $g_{ii} = 0$ as well as all the partial derivatives $\frac{\partial}{\partial g_{ii}} = 0$. We denote by \mathbb{E} expectation with respect to these random variables. To inverse temperature $\beta \in \mathbb{R}$ and external field $h \in \mathbb{R}$, the TAP equations [21] for the SK model [18] are self-consistency equations, which approximately describe the quenched magnetizations $\mathbf{m} = (m_1, \dots, m_N) \in [-1, 1]^N$, i.e. the mean of the Ising spins under Gibbs measure, reading

$$m_i = \tanh \left(h + \frac{\beta}{\sqrt{N}} \sum_{j=1}^N g_{ij} m_j - \beta^2 (1 - q_N) m_i \right) \quad i = 1 \dots N, \quad (1)$$

where

$$q_N \equiv q_N(\mathbf{m}) := \frac{1}{N} \sum_{i=1}^N m_i^2 \quad (2)$$

is the Edwards-Anderson order parameter, EA for short, whereas the "reaction term"

$$\beta^2 (1 - q_N) m_i =: \text{On}_i \quad (3)$$

is the finite-size Onsager correction. These equations are known to be a good approximation for the magnetizations in some high temperature regime [1, 8, 19, 20] and believed to be such at least if β, h satisfy the AT condition

$$\beta^2 \mathbb{E} \frac{1}{\cosh^4(h + \beta \sqrt{q} Z)} \leq 1, \quad (4)$$

where $q \geq 0$ solves

$$q = \mathbb{E} \tanh^2(h + \beta \sqrt{q} Z) =: \phi(q). \quad (5)$$

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We refer to q as the limiting fixed point, but it is known as the replica symmetric solution also; here and henceforth, we denote by Z a standard Gaussian, and by \mathbb{E} its expectation. From now on, q denotes the solution of the fixed point equation (5) which is unique for $h \neq 0$ or $h = 0, |\beta| \leq 1$, and we choose q to be the unique positive fixed point of (5) for $h = 0, |\beta| > 1$.¹

For $i = 1 \dots N$, we refer to

$$h_i \equiv h_i(\mathbf{m}) := \sum_{j=1}^N g_{ij} m_j, \quad (6)$$

as effective fields. In this note, we propose a new algorithmic construction of the solutions of (1). The key observation towards this goal concerns the "true" nature of the finite-size Onsager correction. Indeed, by Gaussian partial integration (PI for short) and Leibniz rule,

$$\mathbb{E} \left[\frac{\beta}{\sqrt{N}} \sum_{j=1}^N g_{ij} m_j \right] \stackrel{(PI)}{=} \frac{\beta}{\sqrt{N}} \sum_{j=1}^N \mathbb{E} \left[\frac{dm_j}{dg_{ij}} \right] \stackrel{(1)}{=} \mathbb{E} [\beta^2 (1 - q_N) m_i] + R_N, \quad (7)$$

where

$$R_N := \mathbb{E} \left[\frac{\beta^2}{\sqrt{N}} \sum_{j \neq i} (1 - m_j^2) \left\{ \sum_{l \neq j} \frac{g_{lj}}{\sqrt{N}} \frac{dm_l}{dg_{ij}} - \beta \frac{d}{dg_{ij}} \{(1 - q_N) m_j\} \right\} - \frac{\beta^2}{N} (1 - m_i^2) m_i \right]. \quad (8)$$

The first term on the r.h.s. of (7) is the (mean of the) Onsager correction (3): it neatly emerges through one Gaussian PI only. The second term captures the nifty fact that the \mathbf{G} -dependence of the $m_i = m_i(g_{kl}, 1 \leq k < l \leq N)$ actually goes on *ad infinitum*. Notwithstanding, a closer look at (8) suggests that stochastic cancellations may dampen such infinite rebounding. In other words, we expect that R_N contributes to lower orders only. This insight turns out to be correct, cfr. Theorem 9 below, and represents the backbone of the present work. Due to the Gaussian nature of the random sum, a first natural guess is of course a central limit theorem; this will indeed turn out to be the case, see Proposition 5 below. To elaborate precisely we define our iteration as follows. For $i \leq N$ and $i, N, k \in \mathbb{N}$ we consider

$$m_i^{(k+1)} := \tanh \left(h + Y_i^{(k)} \right) \quad (9)$$

with the re-centered effective fields given by

$$Y_i^{(k)} := \frac{\beta}{\sqrt{N}} \sum_{j=1}^N g_{ij} m_j^{(k)} - \text{On}_i^{(k)}, \quad (10)$$

where $\text{On}_i^{(k)}$ is given by either of the following choices

- (I) $\text{On}_i^{(k)} := \beta^2 \left(1 - q_N^{(k)} \right) m_i^{(k-1)},$
- (II) $\text{On}_i^{(k)} := \frac{\beta}{\sqrt{N}} \sum_{j=1}^N \mathfrak{d}_{ij} m_j^{(k)}.$

¹For $h = 0$ and $|\beta| > 1$ there are two solutions of (5) namely 0 and $q > 0$. See Lemma 24 for more details.

Here and henceforth we use the notations

$$q_N^{(k)} := \frac{1}{N} \sum_{i=1}^N m_i^{(k)2} \quad \text{and} \quad \mathfrak{d}_{kl}F := \frac{dF}{dg_{kl}},$$

for sufficiently smooth functions F . Choice (I) is the *classical*, finite-size Onsager correction to the TAP equations (1). For this choice the iteration is known to approximate the magnetizations of the SK-Model well, at least in some temperature regime, see [10] for details. By the heuristics related to (7), the second choice (II) rather highlights the Onsager reaction term as a re-centering of the effective fields. All our results are valid in both cases, yet the second (and apparently: new) correction – as well as its interpretation – are arguably more natural within our Stein framework. To properly define $\mathbf{m}^{(k)}$ a starting values $\mathbf{m}^{(1)}$ or in case (I) $\mathbf{m}^{(0)}, \mathbf{m}^{(1)}$ are needed. Concerning these values we assume throughout, that :

A1) The starting value $\mathbf{m}^{(1)} \in [-1, 1]^N$ is either deterministic or independent of the \mathbf{G} -disorder.

A2) In case of choice (I), we additionally set $\mathbf{m}^{(0)} := 0$.

The extent to which A1) can be relaxed is discussed in Remark 10 below.

Clearly the limit of this algorithm, if it converges, is a solution of (1). This motivates our first result, the following *pseudo-convergence*² :

Proposition 1. *For $\beta, h \in \mathbb{R}$ and $\liminf_{N \rightarrow \infty} q_N^{(1)} > 0$ if $h = 0, |\beta| > 1$, we have*

$$\lim_{k, l \rightarrow \infty} \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \left[\left(m_i^{(k)} - m_i^{(l)} \right)^2 \right] = 0 \quad (11)$$

if and only if the AT condition (4) is satisfied.

To investigate further we shorten

$$q_N^{(k, k')} := N^{-1} \sum_i m_i^{(k)} m_i^{(k')} \quad \text{and} \quad q_N^{(k)} := q_N^{(k, k)}. \quad (12)$$

The convergence in (11) goes as follow: We can rewrite the l.h.s. of (11) as

$$\frac{1}{N} \sum_{i=1}^N \left[\left(m_i^{(k)} - m_i^{(l)} \right)^2 \right] = q_N^{(k)} + q_N^{(l)} - 2q_N^{(k, l)}. \quad (13)$$

We will see that the two first terms of the r.h.s. tend to q as $N \rightarrow \infty$. However, $q_N^{(k, l)}$ tends to q as $N \rightarrow \infty$ and $k \rightarrow \infty$ if and only if the AT-condition (4) is satisfied and converges to a limit strictly smaller than q if the AT-condition (4) is not satisfied.

To describe the situation precisely let

$$\psi(t, t', t'') := \mathbb{E} \left[\tanh(h + \beta Z') \tanh(h + \beta Z'') \right] \quad (14)$$

for $t', t'' \in [0, 1]$, $|t| \leq \sqrt{t' t''}$, where the expectation is over centered Gaussians Z', Z'' such that

$$\text{Var}(Z') = t', \quad \text{Var}(Z'') = t'', \quad \text{Cov}(Z', Z'') = t. \quad (15)$$

²This is quite a delicate issue. Indeed, even the meaning of "solution" and "convergence" are *a priori* by far not obvious: due to the mean field character of the random system, one has to first send $N \rightarrow \infty$, and only in a second step $k \rightarrow \infty$. Naturally, the proper procedure to allow for a discussion of the true TAP-solutions would send $k \rightarrow \infty$ with N fixed.

Additionally we say $X_{N,k} \rightarrow X$ in probability as $N \rightarrow \infty$ followed by $k \rightarrow \infty$, if for all $\varepsilon > 0$

$$\lim_{k \rightarrow \infty} \limsup_{N \rightarrow \infty} \mathbb{P}(|X_{N,k} - X| > \varepsilon) = 0.$$

In the same way, we say $X_{N,k} \rightarrow X$ weakly as $N \rightarrow \infty$ followed by $k \rightarrow \infty$, if

$$\lim_{k \rightarrow \infty} \limsup_{N \rightarrow \infty} \mathbb{E}(f(X_{N,k})) = \mathbb{E}(f(X))$$

for all continuous and bounded functions f . Similarly, for $k_1, k_2 \dots k_L \rightarrow \infty$ the $\lim_{k \rightarrow \infty}$ is replaced by $\lim_{k_1, k_2 \dots k_L \rightarrow \infty}$.

Remark 2. *The use of limsup is required to account for arbitrary starting values. It can be replaced by the normal limit if the initial value is well behaved, i.e if $q_N^{(1)}$ and $\frac{1}{N} \sum_{i=1}^N m_i^{(k)}$ converge as $N \rightarrow \infty$.*

Theorem 3. *For all $\beta, h \in \mathbb{R}$ and assuming $\liminf_{N \rightarrow \infty} q_1^{(N)} > 0$ a.s. additionally if $h = 0, |\beta| > 1$, we have*

$$q_N^{(k,k')} \rightarrow \begin{cases} q & \text{if } k = k' \\ \tilde{q} & \text{if } k \neq k' \end{cases} \quad \text{in probability as } N \rightarrow \infty \text{ followed by } k, k' \rightarrow \infty \quad (16)$$

where \tilde{q} is the smallest non-negative fixed point of $f(t) = \psi(t, q, q)$. Additionally $q = \tilde{q}$ if the AT condition (4) is satisfied and $q > \tilde{q}$ otherwise.

Remark 4. *For $h = 0$, the system can not entirely forget the initial condition in the double limit in low temperature. Indeed, if $h = 0, |\beta| > 1$ and $\lim_{N \rightarrow \infty} q_N^{(1)} = 0$, then $q > 0$, but $q_N^{(k,k')} \rightarrow 0$ in probability as $N \rightarrow \infty$. This case is due to the fact that $t \rightarrow \psi(t, t, t) = \phi(t)$ has two fixed points for $h = 0, |\beta| > 1$ (see equation (5) and the comments below): A repulsive fixed point 0 and an attractive fixed point q . Therefore, 0 is only relevant, when the algorithm starts "close" to it.*

Proof of Proposition 1. The claim follows immediately by applying Theorem 3 to (13). \square

We now state the aforementioned CLT for the centered effective fields, which concludes a first big picture overview over the behaviour of the iteration.

Proposition 5. *For all $\beta, h \in \mathbb{R}, L \in \mathbb{N}, (k_l, i_l)_{l \leq L} \in (\mathbb{N} \times \{1, \dots, N\})^L$, and if $h = 0, |\beta| > 1$ assuming $\liminf_{N \rightarrow \infty} q_1^{(N)} > 0$ a.s. additionally, we have*

$$(Y_{i_1}^{(k_1)}, \dots, Y_{i_L}^{(k_L)}) \rightarrow \mathcal{Z},$$

in distribution as $N \rightarrow \infty$ followed by $k_1, \dots, k_L \rightarrow \infty$, such that $(i_l, k_l) \neq (i_{l'}, k_{l'})$ for all $l \neq l'$ and where \mathcal{Z} is a centered Gaussian field with covariance

$$\Sigma_\infty(l, l') := \mathbb{E}[\mathcal{Z}_l \mathcal{Z}_{l'}] = \begin{cases} \beta^2 q & \text{if } l = l' \\ \beta^2 \tilde{q} & \text{if } l \neq l', i_l = i_{l'} \\ 0 & \text{otherwise} \end{cases}$$

Note that this implies that $(Y_1^{(k_1)}, Y_2^{(k_2)}, \dots, Y_L^{(k_L)})$ tends weakly to a vector of independent centered Gaussians of variance $\beta^2 q$. Correlations are only present in the limit between effective fields at the same site, i.e. between $Y_i^{(k)}$ with the same i . If the AT condition (4) is satisfied $q = \tilde{q}$ by Theorem 3, which corresponds to the re-centered effective fields being

perfectly correlated and therefore aligning in the double limit. If however the AT condition is not satisfied, we observe $\tilde{q} < q$, hence in the double limit the centred effective fields are not perfectly correlated. They share a macroscopic Gaussian as a common trunk, but retain an also macroscopic independent Gaussian part, that is reshuffled each iteration.

We will see through Stein's method that the weak convergence of the re-centred effective fields give a new perspective on the AMP algorithms: In fact, the observed weak convergence and law of large numbers are two sides of the same coin. To explain this further we proceed with a high level sketch in the next section. Thereafter we discuss and compare to previous work on the topic in Section 3. We then proceed to proof all results step by step: discussing the approximate Gaussianity of effective fields by Stein's method in section 4 and a law of large numbers for covariances in Section 5. This leads to a deterministic propagation of variances and covariances in the N -limit, which requires a fixed point and contractivity analysis of ϕ and $t \rightarrow \psi(t, q, q)$ discussed in Section 6 to conclude proofs in Section 7.

2. SKETCH OF THE PROOF AND THE IMAGE

In order to prove convergence of the algorithm (9) we develop a treatment which radically differs from [5, 6]: in a first step we will establish, by Stein's method, optimal N -bounds on the distance between the law of the random vector $(Y_i^{(k)})_{(i,k) \in I}$ for some fixed finite $I \subset \mathbb{N} \times \mathbb{N}$ and a "conditional" multivariate Gaussian, cfr. Theorem 6 below. The Gaussian vector is centered with entries of variance $\beta^2 q_N^{(k)}$, which are independent for different sites (values of i).

By exchangeability, we will then deduce that

$$q_N^{(k+1)} \equiv \frac{1}{N} \sum_{i=1}^N \tanh \left(h + Y_i^{(k)} \right)^2 \approx \mathbb{E} \left[\tanh \left(h + \beta \sqrt{q_N^{(k)}} Z \right)^2 \middle| q_N^{(k)} \right] \equiv \phi \left(q_N^{(k)} \right). \quad (17)$$

In a third and final step we show contractivity of the ϕ -function for any β, h , see Theorem 23. The convergence of the algorithm will then follow from the convergence of $q_N^{(k,k')} \rightarrow q$ when N and then $k, k' \rightarrow \infty$ under the AT condition (4).

For $L \in \mathbb{N}$ and any choice of sites and iteration numbers $i_1, \dots, i_L, k_1, \dots, k_L \in \mathbb{N}$ and $N \geq \max_{l \leq L} i_l$, consider the random vector

$$\mathbf{Y} := (Y_{i_l}^{(k_l)})_{l \leq L}. \quad (18)$$

For $r \in \mathbb{N}$ we define the following spaces of test functions

$$\mathcal{H}^r := \{f \in \mathcal{C}^{r+1}(\mathbb{R}^L, \mathbb{R}) : |\partial_\alpha f|_\infty \leq 1 \text{ for } |\alpha| \in \{1, \dots, r\}, |\partial_\alpha f|_\infty < \infty \text{ for } |\alpha| = r+1\}, \quad (19)$$

where α goes over every $|\alpha|$ -fold partial derivative (see (41), (42) for details). With this we define the associated distances between the laws of integrable random variables on \mathbb{R}^L

$$d_r(X, X') = \sup_{f \in \mathcal{H}^r} |\mathbb{E}[f(X)] - \mathbb{E}[f(X')]|. \quad (20)$$

Integral probability metrics of this kind are well known and have been studied in e.g. [12] and the references therein. We also consider a random vector

$$\mathbf{Z} = (Z_{i_l}^{(k_l)})_{l \leq L}, \quad (21)$$

which, *conditionally* on the q_N^{\cdot} -s, is a multivariate centered Gaussian with covariance

$$\Sigma(l, l') := \mathbb{E} \left[Z_{i_l}^{(k_l)} Z_{i_{l'}}^{(k_{l'})} \right] = \begin{cases} \beta^2 q_N^{(k_l, k_{l'})} & \text{if } i_l = i_{l'}, \\ 0 & \text{otherwise,} \end{cases} \quad l, l' = 1 \dots L. \quad (22)$$

Theorem 6. (*Proximity of the effective fields to a Gaussian*). For all $\beta, h \in \mathbb{R}$ and any $L \in \mathbb{N}$, $(k_l, i_l)_{l \leq L} \in (\mathbb{N} \times \{1, \dots, N\})^L$, there exists $C \equiv C(k, L, \beta, h) > 0$ such that

$$d_2(\mathbf{Y}, \mathbf{Z}) \leq \frac{C}{\sqrt{N}}, \quad \forall N \in \mathbb{N}. \quad (23)$$

The above statement is in itself not conclusive: We still have to prove that the q -s, the scalar products of the iterates, see (12), appearing in the covariance (22) converge in the large- N, k limit. Theorem 7 below is the key step towards this fact. We will see that we have an inductive relation between the q_N^{\cdot} -s: Intuitively, it is due to the fact that

$$\begin{aligned} q_N^{(k+1, k'+1)} &\approx \mathbb{E} q_N^{(k+1, k'+1)} = \sum_{i=1}^N \frac{1}{N} \mathbb{E} \left[m_i^{(k+1)} m_i^{(k'+1)} \right] = \mathbb{E} \left[\tanh \left(h + Y_1^{(k)} \right) \tanh \left(h + Y_1^{(k')} \right) \right] \\ &\approx \mathbb{E} \left[\tanh \left(h + Z_1^{(k)} \right) \tanh \left(h + Z_1^{(k')} \right) \right] = \mathbb{E} \psi \left(q_N^{(k, k')}, q_N^{(k)}, q_N^{(k')} \right) \approx \psi \left(q_N^{(k, k')}, q_N^{(k)}, q_N^{(k')} \right), \end{aligned} \quad (24)$$

the first and last equality by concentration and the first line to the second line by Theorem 6. The next theorem makes this precise. To state it we define for $t \geq 0$

$$\chi(t) := \mathbb{E} \left[\tanh(h + \beta \sqrt{t} Z) \right]. \quad (25)$$

Theorem 7. (*Propagation of the covariance entries*). For any $\beta, h \in \mathbb{R}$, and $k, k' \geq 1$ there exists $C \equiv C(k, k', \beta, h) > 0$ such that:

$$\mathbb{E} \left[\left(q_N^{(k+1, k'+1)} - \psi \left(q_N^{(k, k')}, q_N^{(k)}, q_N^{(k')} \right) \right)^2 \right] \leq \frac{C}{\sqrt{N}} \quad (26)$$

and

$$\mathbb{E} \left[\left(q_N^{(k+1, 1)} - \chi \left(q_N^{(k)} \right) \frac{1}{N} \sum_{i=1}^N m_i^{(1)} \right)^2 \right] \leq \frac{C}{\sqrt{N}}. \quad (27)$$

In case that $q_N^{(1)}$ and $\frac{1}{N} \sum_{i=1}^N m_i^{(1)}$ converge as $N \rightarrow \infty$, recursive application of Theorem 7 implies that also $q_N^{(k, k')}$ has a limit in probability as $N \rightarrow \infty$. By analysing the $k, k' \rightarrow \infty$ limit, we obtain the convergence of $q_N^{(k, k')}$ from Theorem 3, which in turn seamlessly yields the convergence of the algorithm: see Proposition 1.

What thus stands behind our results, and is fully captured by the Stein's method, is the following beautiful picture: Provided the mean and radius of the starting magnetisations converge we have for any choice of parameters (inverse of temperature and external field) that any fixed family of effective fields converges weakly, in the infinite volume limit ($N \rightarrow \infty$), to a Gaussian field with covariances depending deterministically on limiting starting radius and mean. This is due to the following interplay: the Gaussian convergence (see Theorem 6) of re-centered effective fields implies a law of large numbers (see Theorem 7) for the limiting covariance structure of re-centered effective fields one iteration further. This procedure explains, how the highly non linear TAP-transformation leaves

finitely many effective fields Gaussian and propagates the starting values only through evolution of the covariance: perhaps surprisingly, this is true in any (β, h) -phase.

Even more interesting is the limiting behaviour when the second limit is taken, that of the iteration parameters: Here the picture does become (highly) parameter dependent as well as fully independent of the starting conditions. This behaviour emerges as iterating the propagation of the covariance structure approaches an attractive fixed point, which turns out to be unique in all cases. This explains the vanishing dependence on starting conditions³. See Theorem 3 and Proposition 5. The temperature dependence crucially comes into play when analysing the fixed point. The system stabilises around a covariance \tilde{q} . If the AT condition is satisfied we observe $\tilde{q} = q$, which means that the effective fields from iteration to iteration macroscopically align, i.e. we obtain approximate solutions of the TAP equation. If the AT condition is not satisfied we observe $\tilde{q} < q$ hence the re-centered effective fields, partially align around a singular Gaussian vector shared among all iterations, but they retain some macroscopic randomness, which is independently reshuffled each iteration. See Theorem 3. We now remark on the special case of vanishing external field which exhibits some special behaviour:

- for vanishing external field and under the AT condition, i.e. $h = 0$ and $|\beta| \leq 1$, the iterates converge towards a degenerate Gaussian field of vanishing variance; this is due to the variance propagation ϕ having 0 as a fixed point for $h = 0$, which is unique and attractive iff $|\beta| \leq 1$: indeed the variances of the effective fields turn out to be approximate solutions of the limiting fixed point equation for the order parameter of the theory (the q -function, in this case a scalar), and indeed converge upon iteration to the unique fixed point. As in this case the unique solution is zero; everything collapses to this trivial limit.
- for vanishing external field and $|\beta| > 1$ (in low temperature) the variances of the limiting Gaussian field still converge to the same fixed point, and so do the covariances (!). Yet, and crucially: the limiting variance now is positive⁴, while the covariances remain zero. Hence the effective fields are fully reshuffled each iteration, leaving the iteration chaotic permanently.

Our proof via Stein thus unravels multiple aspects of the intricate behaviour of mean field spin glasses, in particular those pertaining to the delicate interplay of almost sure phenomena (such is the convergence of the elements of the covariance matrix of the limiting Gaussian fields) and weak limits (of the effective fields). As a matter of fact, it is impossible to say which of these phenomena comes first : things are deeply intertwined, and can hardly be disentangled.

As a final remark we want to mention, that this new method does not really depend on the Gaussian nature of the disorder and the implementation of this method for other mean fields models is left to future research.

3. PREVIOUS WORK: THE BOLTHAUSEN'S ALGORITHMUS

An algorithmic construction of the solutions to a *modified* version of (1) has been obtained by Bolthausen [5, 6] (see also [10] for another iterative scheme), who studied the

³except for $h = 0$ and $m^{(1)} \approx 0$

⁴except for $m^{(1)} \approx 0$

approximate version of TAP equations

$$m_i = \tanh \left(h + \frac{\beta}{\sqrt{N}} \sum_{j=1}^N g_{ij} m_j - \beta^2 (1-q) m_i \right) \quad i = 1 \dots N, \quad (28)$$

where the EA-parameter q_N from (1) is replaced by a scalar⁵ q . To avoid confusion, we shall refer to the *modified* TAP equations (28) as B-TAP. Bolthausen's algorithm for B-TAP is initialised with the deterministic values

$$\mathbf{m}_i^{(0)} := 0, \quad \mathbf{m}_i^{(1)} := \sqrt{q}, \quad \forall i \leq N \quad (29)$$

which are then propagated by a two step Banach procedure ($k-1, k \rightarrow k+1$) given by

$$m_i^{(k+1)} = \tanh \left(h + \frac{\beta}{\sqrt{N}} \sum_{j=1}^N g_{ij} m_j^{(k)} - \beta^2 (1-q) m_i^{(k-1)} \right), \quad k \geq 1. \quad (30)$$

For $h \neq 0$, it is shown in the seminal paper [5] via some delicate analysis involving Gaussian projections, conditioning techniques, and law of large numbers that the following *pseudo-convergence* holds:

$$\lim_{k,l \rightarrow \infty} \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \mathbb{E} \left[\left(m_i^{(k)} - m_i^{(l)} \right)^2 \right] = 0, \quad (31)$$

if and only if (β, h) satisfy the AT condition (4).

The above treatment, to our knowledge, is the only procedure available so far for a rigorous analysis of B-TAP up to the AT line, but suffers from a number of shortcomings.

- First of all, the Onsager correction: by replacing q_N , the random and finite-size EA-order parameter, with q , the deterministic solution to (5) one artificially injects a posteriori information into the model a priori: It does not allow to understand whether the expected behaviour emerges by itself or was built in by this choice of q and starting point. Indeed, by work of Plefka [17], the TAP equations are known to be the critical points of the expansion of the Gibbs potential to second order, but the Onsager correction (28) is not compatible with Plefka's framework nor with the original TAP derivation [21], see also [13, 14] for more on this (and related) issue(s).
- A second delicate point is the initialisation (29): the B-TAP-algorithm is known to converge for this very specific choice of starting value only, somewhat corresponding to the odd situation where one solves a fixed point equation by starting virtually 'in' the solution. Since the procedure is but a sophisticated Banach-iteration, this restriction is unnatural, and hides key phenomena concerning the onset of the spin glass phase.
- A third point is that the Bolthausen algorithm does not allow to investigate the case $h = 0$.
- Finally we want to make the point, that the sophisticated conditioning technique developed by Bolthausen seems to be dependent on the Gaussian nature of the disorder, but certainly can only work if at least the effective fields are approximately Gaussian. The method presented here is more robust in the sense that approximate integration by parts and Stein operators for different distributions are established tools, which allow approaching these scenarios.

⁵Corresponding, albeit only *a posteriori*, to the \mathbb{P} -almost sure q_N -limit.

Nevertheless these magnetizations, provided by the pseudo-convergence, allowed Bolthausen [6] to give a proof of the replica symmetric formula for the free energy of the SK model in high temperature. Brennecke and Yau [7] proved an extension of this result. However, neither result is valid for all (β, h) satisfying the AT condition (4), which is believed to separate the high-temperature from the low-temperature regime. Extending aforementioned results up to AT is an open problem.

Bolthausen's algorithm and conditioning techniques have found wide applications, in particular they have been used by Montanari and coauthors to analyse Approximate Message Passing (AMP for short) in a large class of models, see [2, 3, 4] and references therein, see also [22] for a theoretical physics' take. As our approach does not require Gaussian disorder we believe that this new method is applicable to an even wider range of models and problems. Also considering the possibility of adapting Stein's method e.g. to the diluted case where the limiting distribution of effective fields is expected to be non Gaussian, by considering an appropriate Stein operator.

4. STEIN TAP

4.1. Overview. The main objective of this section is to prove Theorem 6 using Stein equation. Before we tackle this task we prove Proposition 5 from Theorem 3 and 6.

Proof of Proposition 5. Clearly

$$d_2(\mathbf{Y}, \mathcal{Z}) \leq d_2(\mathbf{Y}, \mathbf{Z}) + d_2(\mathbf{Z}, \mathcal{Z}), \quad (32)$$

where the first term tends to 0 as $N \rightarrow \infty$ by Theorem 6. By Theorem 3 Σ converges to Σ_∞ as $N \rightarrow \infty$ followed by $k_1, \dots, k_L \rightarrow \infty$. Now we consider the simple coupling $\mathbf{Z} = \Sigma^{1/2} Z$ and $\mathcal{Z} = \Sigma_\infty^{1/2} Z$, for a standard Gaussian vector Z independent of Σ and use for $f \in \mathcal{H}^2$ the estimate

$$\mathbb{E}|f(\mathcal{Z}) - f(\mathbf{Z})| \leq \|\nabla f\|_2 \mathbb{E}\|\Sigma^{1/2} - \Sigma_\infty^{1/2}\|_{op} \leq L \mathbb{E}\|\Sigma^{1/2} - \Sigma_\infty^{1/2}\|_{op}, \quad (33)$$

Since all operations involved on the r.h.s. are continuous in the entries of Σ we see that the second term of (32) vanishes with $N \rightarrow \infty$ followed by $k_1, \dots, k_L \rightarrow \infty$, since the operator norm involved is bounded:

$$\|\Sigma^{1/2}\|_{op} \leq \|\Sigma^{1/2}\|_F \leq L \|\Sigma^{1/2}\|_\infty \leq L \sqrt{|\text{diag}(\Sigma)|_\infty} \leq L\beta,$$

where $\|\cdot\|_F$ is the Frobenius norm. From (32) vanishing, we obtain the asserted weak convergence, since the vector space of finite linear combinations of \mathcal{H}^2 functions is dense in the continuous functions when restricting to a compact by Stone Weierstrass Theorem. The restriction to compact support is then removed using tightness of both sequences, which holds since the covariance matrices in question have entries bounded uniformly in N . \square

We will work with test functions in \mathcal{H}^2 , since it is suitable to prove weak convergence, includes tanh and matches nicely with the regularity the Stein operator provides. To accommodate for Σ in Theorem 6 being random we slightly modify the classical Stein equation. One easily checks that Σ is in fact symmetric and positive semi-definite, and consequently $\Sigma^{1/2}$ exists and is symmetric. For Z a standard Gaussian on \mathbb{R}^L , $F \in \mathcal{C}^3(\mathbb{R}^L, \mathbb{R})$ with

$|\partial_\alpha F|_\infty < \infty$ for $|\alpha| \in \{1, 2, 3\}$ we define

$$V_F(x) = - \int_0^\infty \mathbb{E}_Z \left[F(xe^{-u} + \sqrt{1 - e^{-2u}}Z) \right] - \mathbb{E}_Z[F(Z)] du.$$

By the multivariate Stein equation (see e.g. Lemma 2.6 in [9]),

$$F(x) - \mathbb{E}_Z F(Z) = \text{Tr} \nabla^2 V_F(x) - x^T \nabla V_F(x). \quad (34)$$

For $f \in \mathcal{H}^2$ we define

$$v(y) = - \int_0^\infty \mathbb{E}_Z \left[f(ye^{-u} + \sqrt{1 - e^{-2u}}\Sigma^{1/2}Z) \right] - \mathbb{E}_Z[f(\Sigma^{1/2}Z)] du,$$

where we recall that \mathbb{E}_Z is the expectation only over Z , which is chosen to be independent of everything else, i.e. of the (g_{ij}) and $m^{(1)}$, hence also of Σ .

Observe that by choosing

$$F(y) = f(\Sigma^{1/2}y), \quad \text{verifying} \quad V_F(y) = v(\Sigma^{1/2}y)$$

and plugging into (34) with $x = \Sigma^{1/2}y$, we obtain

$$f(x) - \mathbb{E}_Z f(\Sigma^{1/2}Z) = \text{Tr}(\Sigma \nabla^2 v(x)) - x^T \nabla v(x). \quad (35)$$

Now note that for $|\alpha| \in \{1, 2, 3\}$

$$\partial_\alpha v(y) = - \int_0^\infty e^{-u|\alpha|} \mathbb{E}_Z \left[\partial_\alpha f(ye^{-u} + \sqrt{1 - e^{-2u}}\Sigma^{1/2}Z) \right] du, \quad (36)$$

which gives us $v \in \mathcal{H}^2$, since $f \in \mathcal{H}^2$ by assumption and

$$|\partial_\alpha v|_\infty \leq \frac{1}{|\alpha|} |\partial_\alpha f|_\infty. \quad (37)$$

Using $x = (Y_{i_l}^{(k_l)})_{l \leq L}$ in (35), writing $v_l = \partial_{e_l} v$ and taking expectations we have

$$\begin{aligned} & \mathbb{E} f \left((Y_{i_j}^{(k_j)})_{j \leq L} \right) - \mathbb{E} f \left(Z_{\Sigma^{(i,j)}} \right) \\ &= \mathbb{E} \left[\text{Tr} \left(\Sigma \nabla^2 v \left((Y_{i_j}^{(k_j)})_{j \leq L} \right) \right) \right] - \mathbb{E} \left[(Y_{i_j}^{(k_j)})_{j \leq L}^T \nabla v \left((Y_{i_j}^{(k_j)})_{j \leq L} \right) \right] \\ &= \sum_{l=1}^L \mathbb{E} \left[\beta^2 \sum_{l'=1}^L \delta_{i_l, i_{l'}} \frac{(m^{(k_l)})^T m^{(k_{l'})}}{N} \partial_{e_{l'}} v_l \right] - \mathbb{E} \left[Y_{i_l}^{(k_l)} v_l \right] \end{aligned} \quad (38)$$

(39)

where we simplify notation by suppressing the argument of v_l , which is $(Y_{i_j}^{(k_j)})_{j \leq L}$. In order to prove Theorem 6, it remains to show that the r.h.s. is bounded in absolute value by C/\sqrt{N} . To this end we need to control the following quantities:

$$\Delta_{x;y,z}^{(k)} := \frac{\beta}{\sqrt{N}} \sum_l g_{x,l} \partial_{yz} m_l^{(k)} - \partial_{yz} \text{On}_x^{(k)}, \quad \mathcal{E}_x^{(k)} := \frac{\beta}{\sqrt{N}} \sum_{l \neq x} \partial_{xl} m_l^{(k)} - \text{On}_x^{(k)}. \quad (40)$$

This is the object of study of our next Theorem, which is shown in Section 4.3.

Theorem 8. For $\beta \in \mathbb{R}$, $k \in \mathbb{N}$ there exists $C = C_{k,\beta} > 0$, s.t.

$$\mathbb{E} \left[\left(\Delta_{x;y,z}^{(k)} \right)^2 \right] \leq \frac{C}{N^2} \quad \text{and} \quad \mathbb{E} \left[\left(\mathcal{E}_x^{(k)} \right)^2 \right] \leq \frac{C}{N}$$

for all $h \in \mathbb{R}$, $N \in \mathbb{N}$ and $x, y, z \leq N$ with $y \neq z$.

To derive Theorem 8, a fundamental understanding of the magnitude of derivatives of m with respect to g is required. To this end, we introduce some notation. First we use multiindex notation, i.e. for

$$\alpha \in \mathbb{N}_0^{\mathfrak{p}}, \quad \text{where} \quad \mathfrak{p} \equiv \mathfrak{p}_N := \{\{i, j\} : 1 \leq i < j \leq N\} \quad (41)$$

we use the shorthands

$$|\alpha| := \sum_{x \in \mathfrak{p}} \alpha_x, \quad \alpha! := \prod_{x \in \mathfrak{p}} \alpha_x! \quad \text{and} \quad \partial_\alpha := \prod_{x \in \mathfrak{p}} (\partial_x)^{\alpha_x}, \quad (42)$$

also recalling the convention $\mathfrak{d}_{ij} = \frac{d}{dg_{ij}}$. Additionally, for any $e \in \mathfrak{p}$ we write $\delta_e \in \mathbb{N}_0^{\mathfrak{p}}$ to refer to the vector with a 1 in the e -coordinate and otherwise zeros. Finally, we associate to each multiindex $\alpha \in \mathbb{N}_0^{\mathfrak{p}}$ the undirected multigraph on $V = \{i : \exists j : \alpha_{\{i,j\}} > 0\}$ with $\alpha_{\{i,j\}}$ edges between vertices i and j . This allows us to use α as a multiindex and its associated multigraph interchangeably. Consequently $i \in \alpha$ means $\exists j : \alpha_{\{i,j\}} > 0$ and $\{i, j\} \in \alpha$ means $\alpha_{\{i,j\}} > 0$. The following key observation is proved in Section 4.2.

Theorem 9. *For $\beta \in \mathbb{R}$, $t, k \in \mathbb{N}$ there exists $C \equiv C_{|\alpha|, t, k, \beta} > 0$, such that*

$$\mathbb{E} \left[\left(\sqrt{N}^{|\alpha| + \mathcal{C}_\alpha(p)} \partial_\alpha m_p^{(k)} \right)^{2t} \right] \leq C \quad (43)$$

for all $\alpha \in \mathbb{N}_0^{\mathfrak{p}}$, $N \in \mathbb{N}$, $p \leq N$, $h \in \mathbb{R}$ and where $\mathcal{C}_\alpha(p)$ denotes the number of connected components in α that do not contain p .

In the applications, we only use the case $t = 1$. However, the proof of this Theorem is by induction on k and requires all t .

Remark 10. *By taking conditional expectations, the starting values $m^{(0)}$, $m^{(1)}$ can be replaced by random ones that are independent of g (the constants in our results do not depend on $m^{(0)}$, $m^{(1)}$). As the proof shows, Theorem 9 also holds if the starting values $m^{(0)}$, $m^{(1)}$ of the iteration depend on g as long as they satisfy (43). The assumption that $m^{(0)}$ and $m^{(1)}$ are independent of g becomes crucial in the proof of (27) in Theorem 7.*

Building on Theorems 8 and 9, we now infer:

Proof of Theorem 6 (using Theorems 8 and 9). It remains to show that the r.h.s. of (38) is bounded in absolute value by C/\sqrt{N} , which we do for each summand, i.e. each l . Using Gaussian integration by parts, the representation (36) of v_l , and the definitions (40), we obtain

$$\mathbb{E} \left[Y_{i_l}^{(k_l)} v_l \right] = \mathbb{E} \left[\mathcal{E}_{i_l}^{(k_l)} v_l \right] + \mathbb{E} \left[\sum_{j \leq L} \frac{\beta}{\sqrt{N}} \sum_{a \neq i_l} m_a^{(k_l)} (\mathfrak{d}_{i_l a} Y_{i_j}^{(k_j)}) \partial_{e_j} v_l \right] + \mathbb{E} \left[\sum_{j \leq L} \frac{\beta}{\sqrt{N}} \sum_{a \neq i_l} m_a^{(k_l)} \mathcal{F}_{i_l a}^j \right] \quad (44)$$

where

$$\mathcal{F}_{i_l a}^j = - \int_0^\infty e^{-u} \sqrt{1 - e^{-2u}} \mathbb{E}_Z \left[((\mathfrak{d}_{i_l a} \Sigma^{1/2}) Z)_j \partial_{e_j + e_l} f(\mathbf{Y} e^{-u} + \sqrt{1 - e^{-2u}} \Sigma^{1/2} Z) \right] du \quad (45)$$

and $\mathbf{Y} = (Y_{i_l}^{(k_j)})_{j \leq L}$.

By simply using that $(\mathbb{E}X)^2 \leq \mathbb{E}(X^2)$, we get that the last expectation in (44) has absolute value bounded by the square root of

$$\begin{aligned} \frac{\beta^2}{N} \sum_{a,a' \neq i_l} \mathbb{E} \left[\left(\sum_{j \leq L} m_a^{(k_l)} \mathcal{F}_{i_l a}^j \right) \left(\sum_{j' \leq L} m_{a'}^{(k_l)} \mathcal{F}_{i_l a'}^{j'} \right) \right] \\ = \frac{\beta^2}{N} \sum_{a,a' \neq i_l} \mathbb{E} \left[m_a^{(k_l)} m_{a'}^{(k_l)} \left(\sum_{j \leq L} \mathcal{F}_{i_l a}^j \right) \left(\sum_{j' \leq L} \mathcal{F}_{i_l a'}^{j'} \right) \right], \end{aligned}$$

which in turn is bounded by

$$\leq \frac{\beta^2}{N} \sum_{a,a' \neq i_l} \mathbb{E} \left[\left(\sum_{j \leq L} \mathcal{F}_{i_l a}^j \right)^2 \right]^{1/2} \mathbb{E} \left[\left(\sum_{j' \leq L} \mathcal{F}_{i_l a'}^{j'} \right)^2 \right]^{1/2} \leq C_{\beta,k,L} N^{-1}$$

by Cauchy-Schwarz, and the following Lemma, which is shown at the end of this proof.

Lemma 11. *Let $\mathcal{F}_{i_l a}^j$ be defined by (45). Then, there exists a constant $C = C_{\beta,k,L}$ such that*

$$\mathbb{E} \left[\left(\sum_{j=1}^L \mathcal{F}_{i_l a}^j \right)^2 \right] \leq C N^{-2}.$$

We thus conclude that

$$\left| \mathbb{E} \left[\sum_{j \leq L} \frac{\beta}{\sqrt{N}} \sum_{a \neq i_l} m_a^{(k_l)} \mathcal{F}_{i_l a}^j \right] \right| \leq C_{\beta,k,L} N^{-1/2} \quad (46)$$

To compute the second summand in (44), we consider derivatives of Y , which are given by

$$\partial_{yz} Y_x^{(k)} = \frac{\beta}{\sqrt{N}} \sum_{b \neq x} \delta_{\{b,x\}\{y,z\}} m_b^{(k)} + \Delta_{x;y,z}^{(k)}. \quad (47)$$

Hence we have

$$\begin{aligned} \frac{\beta}{\sqrt{N}} \sum_{a \neq i_l} m_a^{(k_l)} \partial_{i_l a} Y_{i_j}^{(k_j)} &= \frac{\beta^2}{N} \sum_{a \neq i_l, b \neq i_j} \delta_{\{b,i_j\}\{i_l,a\}} m_a^{(k_l)} m_b^{(k_j)} + \frac{\beta}{\sqrt{N}} \sum_{a \neq i_l} m_a^{(k_l)} \Delta_{i_j; i_l, a}^{(k_j)} \\ &= \beta^2 \delta_{i_l i_j} q_N^{(k_l, k_j)} + \frac{\beta}{\sqrt{N}} \sum_{a \neq i_l} m_a^{(k_l)} \Delta_{i_j; i_l, a}^{(k_j)} + O(N^{-1}), \end{aligned} \quad (48)$$

where the $O(N^{-1})$ terms is only due to a difference of form $\pm N^{-1} \beta^2 m_a^{(k_l)} m_b^{(k_j)}$. Plugging (48) into (44) and this into the r.h.s. of (38) we see that the $\beta^2 \delta_{i_l i_j} q_N^{(k_l, k_j)}$ term in (48) cancels with the first part of each summand on the r.h.s of (38). Hence up error of size at most $C_{\beta,k,L}/\sqrt{N}$ by (46), the l -th summand on the r.h.s. of (38) equals

$$\mathbb{E}[\mathcal{E}_{i_l}^{(k_l)} v_l] + \sum_{l'=1}^L \mathbb{E} \left[\frac{\beta}{\sqrt{N}} \sum_{a \neq i_l} m_a^{(k_l)} \Delta_{i_l'; i_l, a}^{(k_l')} \partial_{e_{l'}} v_l \right]. \quad (49)$$

Since $|v_l|_\infty \leq 1$ by (37) and using Jensen or Cauchy-Schwarz the first expectation is at most $C_{\beta,k} N^{-1/2}$ by Theorem 8. Similarly estimating the second expectation using

$|m|, |\partial_{e_l} v_l|_\infty \leq 1$ we obtain

$$\left| \mathbb{E} \left[\frac{\beta}{\sqrt{N}} \sum_{a \neq i_l} m_a^{(k_l)} \Delta_{i_l'; i_l, a}^{(k_l')} \partial_{e_l} v_l \right] \right| \leq \beta \sqrt{N} \max_{a \neq i_l} \mathbb{E} [|\Delta_{i_l'; i_l, a}^{(k_l')}|] \leq C_{\beta, k} \sqrt{N} \max_{a \neq i_l} \sqrt{\mathbb{E} \left[\left(\Delta_{i_l'; i_l, a}^{(k_l')} \right)^2 \right]}.$$

Application of Theorem 8 shows that the r.h.s. in the last display is bounded by $C_{\beta, k} N^{-1/2}$. Hence, the r.h.s. in (38) is bounded by $C_{\beta, k, L} N^{-1/2}$ as asserted. We now come back to the last missing piece, which is the

Proof of Lemma 11. By Gaussian integration by parts,

$$\begin{aligned} \sum_{j=1}^L \mathcal{F}_{i_a}^j &= - \sum_{j=1}^L \int_0^\infty e^{-u} \sqrt{1 - e^{-2u}} \mathbb{E}_Z \left[((\partial_{i_a} \Sigma^{1/2}) Z)_j \partial_{e_j + e_l} f(\mathbf{Y} e^{-u} + \sqrt{1 - e^{-2u}} \Sigma^{1/2} Z) \right] du \\ &= - \int_0^\infty e^{-u} (1 - e^{-2u}) \\ &\quad \cdot \sum_{j, p, t=1}^L \mathbb{E}_Z \left[((\partial_{i_a} \Sigma^{1/2}))_{jp} (\Sigma^{1/2})_{tp} \partial_{e_j + e_l + e_t} f(\mathbf{Y} e^{-u} + \sqrt{1 - e^{-2u}} \Sigma^{1/2} Z) \right] du. \end{aligned} \quad (50)$$

As $\Sigma^{1/2}$ is symmetric we have $\partial_{i_a} \Sigma = \partial_{i_a} (\Sigma^{1/2} \Sigma^{1/2}) = \Sigma^{1/2} \partial_{i_a} \Sigma^{1/2} + (\partial_{i_a} \Sigma^{1/2}) \Sigma^{1/2}$, the sum in the last line equals

$$\begin{aligned} S(u) &:= \mathbb{E}_Z \left[\text{tr} \left((\partial_{i_a} \Sigma^{1/2}) \Sigma^{1/2} (\nabla^2 \partial_{e_l} f)(\mathbf{Y} e^{-u} + \sqrt{1 - e^{-2u}} \Sigma^{1/2} Z) \right) \right] \\ &= \mathbb{E}_Z \left[\text{tr} \left(\Sigma^{1/2} (\partial_{i_a} \Sigma^{1/2}) (\nabla^2 \partial_{e_l} f)(\mathbf{Y} e^{-u} + \sqrt{1 - e^{-2u}} \Sigma^{1/2} Z) \right) \right] \\ &= \mathbb{E}_Z \left[\text{tr} \left(\frac{1}{2} (\partial_{i_a} \Sigma) (\nabla^2 \partial_{e_l} f)(\mathbf{Y} e^{-u} + \sqrt{1 - e^{-2u}} \Sigma^{1/2} Z) \right) \right]. \end{aligned}$$

The middle equality due to the trace being invariant under transposition as well as cyclic permutation of products. By Cauchy-Schwarz, the expression in the last display is less than

$$S(u) \leq \frac{1}{2} \mathbb{E}_Z \left[\sqrt{\text{tr} \left((\partial_{i_a} \Sigma)^2 \right) \text{tr} \left((\nabla^2 \partial_{e_l} f)^2 (\mathbf{Y} e^{-u} + \sqrt{1 - e^{-2u}} \Sigma^{1/2} Z) \right)} \right].$$

As $|\partial_{e_l + e_r + e_s} f|_\infty \leq 1$, it follows that

$$\mathbb{E} [S(u)^2] \leq L^2 \mathbb{E} \left(\text{tr} \left((\partial_{i_a} \Sigma)^2 \right) \right) = L^2 \sum_{r=1}^L \sum_{s=1}^L \mathbb{E} \left((\partial_{i_a} \Sigma_{rs})^2 \right).$$

By (22),

$$(\partial_{i_a} \Sigma_{rs})^2 \leq \left(\partial_{i_a} \frac{\beta^2}{N} \sum_{i=1}^N m_i^{(k_r)} m_i^{(k_s)} \right)^2 = \left(\frac{\beta^2}{N} \sum_{i=1}^N m_i^{(k_r)} \partial_{i_a} m_i^{(k_s)} + \frac{\beta^2}{N} \sum_{i=1}^N m_i^{(k_s)} \partial_{i_a} m_i^{(k_r)} \right)^2.$$

Using the fact that $(a + b)^2 \leq 2(a^2 + b^2)$, we obtain

$$\begin{aligned} \mathbb{E} \left[(\partial_{i_a} \Sigma_{rs})^2 \right] &\leq 2 \frac{\beta^4}{N^2} \sum_{i, j=1}^N \mathbb{E} \left[m_i^{(k_r)} (\partial_{i_a} m_i^{(k_s)}) m_j^{(k_r)} (\partial_{i_a} m_j^{(k_s)}) \right] \\ &\quad + \mathbb{E} \left[m_i^{(k_s)} (\partial_{i_a} m_i^{(k_r)}) m_j^{(k_s)} (\partial_{i_a} m_j^{(k_r)}) \right]. \end{aligned}$$

By Cauchy-Schwarz, the first term on the r.h.s. of the latter is smaller than

$$\leq 2 \frac{\beta^4}{N^2} \sum_{i,j=1}^N \mathbb{E} \left[\left(\partial_{i,a} m_i^{(k_s)} \right)^2 \right]^{1/2} \mathbb{E} \left[\left(\partial_{i,a} m_j^{(k_s)} \right)^2 \right]^{1/2} \leq C_{\beta,k} N^{-2},$$

and exactly the same estimate holds for the second term. The last inequality follows as the second moments in the last display are $\leq C_{\beta,k} N^{-1-1_{i \notin \{i,a\}}}$ and $\leq C_{\beta,k} N^{-1-1_{j \notin \{i,a\}}}$ by Theorem 9. Plugging these estimates into (50) and using Cauchy-Schwarz inequality, we obtain

$$\mathbb{E} \left[\left(\sum_{j=1}^L \mathcal{F}_{i,a}^j \right)^2 \right] = \iint_0^\infty -e^{-u-u'} (1-e^{-2u})(1-e^{-2u'}) \mathbb{E} [S(u)^2]^{1/2} \mathbb{E} [S(u')^2]^{1/2} du du' \leq C_{\beta,k,L} N^{-2}$$

as asserted. \square

\square

\square

4.2. Controlling derivatives / Proof of Theorem 9. In the proof, we use the following consequence of the Hölder inequality which can be shown by induction: For any $n \in \mathbb{N}$ and random variables X_1, \dots, X_n in $L_n(\mathbb{P})$, we have

$$\mathbb{E} [X_1 \cdots X_n]^n \leq \mathbb{E} [|X_1|^n] \cdots \mathbb{E} [|X_n|^n]. \quad (51)$$

Proof of Theorem 9. We show (43) by induction on k . The base clauses, $k = 1$ for choice (II) and $k = 0, 1$ for (I), hold, since the starting values are independent of g . From here on we may assume that the assertion is correct up to $k - 1$, but before this will be helpful some preparation and computation is needed. To this end we write $A \preceq B$ if $A \leq CB$ for a $C > 0$ that depends on $\beta, t, k, |\alpha|$ only and denote by $\tanh^{(s)}$ the s -th derivative of \tanh . By the generalized Faa di Bruno formula (see e.g. [11, Corollary 2.10], [15]), for each $\alpha \in \mathbb{N}_0^{\mathfrak{p}}$ with $|\alpha| \geq 1$,

$$\partial_\alpha m_p^{(k)} = \alpha! \sum_{(\sigma, s) \in \mathcal{P}(\alpha)} \tanh^{(|s|)}(h_p^{(k)}) \prod_\ell \frac{1}{s_\ell!} \left(\frac{\partial_{\sigma_\ell} h_p^{(k)}}{\sigma_\ell!} \right)^{s_\ell}, \quad (52)$$

where

$$\mathcal{P}(\alpha) := \left\{ (\sigma, s) \in \bigcup_{u=1}^{|\alpha|} ((\mathbb{N}_0^{\mathfrak{p}})^u \times \mathbb{N}^u) : \sum_i s_i \sigma_i = \alpha, \quad 0 < \sigma_1 < \dots < \sigma_u \right\}.$$

Note that " $<$ " in this definition can be chosen to be any order on $\mathbb{N}_0^{\mathfrak{p}}$ with minimal element 0. From (52) we immediately obtain the representation

$$\mathbb{E} \left[\left(\partial_\alpha m_p^{(k)} \right)^{2t} \right] = \sum_{(\sigma^{(b)}, s^{(b)})_{b \in \mathcal{P}(\alpha)^{2t}}} \mathbb{E} \left[\prod_{b=1}^{2t} \alpha! \tanh^{(|s^{(b)}|)}(h^{(k)}) \prod_\ell \frac{1}{s_\ell^{(b)}!} \left(\frac{\partial_{\sigma_\ell^{(b)}} h_p^{(k)}}{\sigma_\ell^{(b)}!} \right)^{s_\ell^{(b)}} \right].$$

Roughly estimating all terms independent of N and using that derivatives of \tanh are bounded (since they are polynomials in \tanh) we obtain

$$\begin{aligned}
&\preceq \sum_{(\sigma^{(b)}, s^{(b)})_{b \in \mathcal{P}} \in \mathcal{P}(\alpha)^{2t}} \mathbb{E} \left[\prod_{b=1}^{2t} \prod_{\ell} \left| \partial_{\sigma_{\ell}^{(b)}} h_p^{(k)} \right|^{s_{\ell}^{(b)}} \right] \\
&\leq \sum_{(\sigma^{(b)}, s^{(b)})_{b \in \mathcal{P}} \in \mathcal{P}(\alpha)^{2t}} \prod_{b=1}^{2t} \prod_{\ell} \mathbb{E} \left[\left(\partial_{\sigma_{\ell}^{(b)}} h_p^{(k)} \right)^{2t \dim(s^{(b)})_{s_{\ell}^{(b)}}} \right]^{\frac{1}{2t \dim(s^{(b)})}},
\end{aligned}$$

the last step by the Hölder inequality (51) and dim mapping a vector to its dimension. Finally we estimate by worst case scenario using that the number of summands does not depend on N to obtain

$$\preceq \max_{(\sigma, s) \in \mathcal{P}(\alpha)} \prod_{\ell} \mathbb{E} \left[\left(\partial_{\sigma_{\ell}} h_p^{(k)} \right)^{2t \dim(s)_{s_{\ell}}} \right]^{\frac{1}{\dim(s)}}. \quad (53)$$

Computing derivatives of h from its definition yields

$$\partial_{\gamma} h_p^{(k)} = N^{-1/2} \sum_i g_{p,i} \partial_{\gamma} m_i^{(k-1)} + N^{-1/2} \sum_i \mathbf{1}_{\gamma_{\{p,i\}} > 0} \partial_{\gamma - \delta_{\{p,i\}}} m_i^{(k-1)} + \partial_{\gamma} \text{On}_p^{(k-1)} \quad (54)$$

for any $\gamma \in \mathbb{N}_0^p$ and writing $\delta_{\{p,i\}}$ for the vector that has all zeros except a single one in the $\{p, i\}$ coordinate. We use $\{, \}$ here to indicate that $\{p, i\}$ and $\{i, p\}$ refer to the same coordinate. Also note that by slight abuse of notation if $i = p$ we set the indicator to zero. Combining (54) with the elementary estimate $(a + b + c)^{2n} \leq 3^{2n}(a^{2n} + b^{2n} + c^{2n})$ we obtain for $m \in \mathbb{N}_0, n \leq 1$

$$\begin{aligned}
\mathbb{E} \left[\left(\partial_{\gamma} h_p^{(k)} \right)^{2n} \right] &\leq N^{-n} \mathbb{E} \left[\left(\sum_i g_{p,i} \partial_{\gamma} m_i^{(k-1)} \right)^{2n} \right] \\
&\quad + N^{-n} \mathbb{E} \left[\left(\sum_i \mathbf{1}_{\gamma_{\{p,i\}} > 0} \partial_{\gamma - \delta_{\{p,i\}}} m_i^{(k-1)} \right)^{2n} \right] + \mathbb{E} \left[\left(\partial_{\gamma} \text{On}_p^{(k-1)} \right)^{2n} \right].
\end{aligned}$$

Each of these expectations will be handled separately by the the following Lemmata:

Lemma 12. *Assuming that the assertion of Theorem 9 holds up to $k - 1$ we have for $n, |\gamma| \leq 1$ that*

$$\mathbb{E} \left[\left(\sum_i g_{p,i} \partial_{\gamma} m_i^{(k-1)} \right)^{2n} \right] \leq N^{n(1-|\gamma| - \mathcal{E}_{\gamma}(p))}$$

Lemma 13. *Assuming that the assertion of Theorem 9 holds up to $k - 1$ we have for $n, |\gamma| \leq 1$ that*

$$\mathbb{E} \left[\left(\sum_i \mathbf{1}_{\gamma_{\{p,i\}} > 0} \partial_{\gamma - \delta_{\{p,i\}}} m_i^{(k-1)} \right)^{2n} \right] \leq N^{n(1-|\gamma| - \mathcal{E}_{\gamma}(p))}$$

Lemma 14. *Assuming that the assertion of Theorem 9 holds up to $k - 1$ we have for $n, |\gamma| \leq 1$ that*

$$\mathbb{E} \left[\left(\partial_{\gamma} \text{On}_p^{(k-1)} \right)^{2n} \right] \leq N^{n(-|\gamma| - \mathcal{E}_{\gamma}(p))}$$

Given Lemmata 12, 13, 14 we have

$$\mathbb{E} \left[\left(\partial_\gamma h_p^{(k)} \right)^{2n} \right] \preceq N^{-n(|\gamma| + \mathcal{C}_\gamma(p))}$$

for $n \preceq 1$. Consequently, by (53) and recalling that $\sum_i s_i \sigma_i = \alpha$, the claim follows since

$$\mathbb{E} \left[\left(\partial_\alpha m_p^{(k)} \right)^{2t} \right] \preceq \max_{(\sigma, s) \in \mathcal{P}(\alpha)} \prod_\ell N^{-ts_\ell(|\sigma_\ell| + \mathcal{C}_{\sigma_\ell}(p))} = N^{-t(|\alpha| + \min_{(\sigma, s) \in \mathcal{P}(\alpha)} \sum_\ell s_\ell \mathcal{C}_{\sigma_\ell}(p))}.$$

This concludes the proof, as $\sum_\ell s_\ell \mathcal{C}_{\sigma_\ell}(p) \geq \mathcal{C}_\alpha(p)$ for $(\sigma, s) \in \mathcal{P}(\alpha)$, where the inequality holds as decomposition into subgraphs can only increase the number of connected components. \square

We now return to the proofs for Lemmata 12, 13, 14.

Proof of Lemma 12. Since

$$\mathbb{E} \left[\left(\sum_{i=1}^N g_{p,i} \partial_\gamma m_i^{(k-1)} \right)^{2n} \right] = \sum_{i_1, \dots, i_{2n}=1}^N \mathbb{E} \left[\prod_{l=1}^{2n} g_{p, i_l} \partial_\gamma m_{i_l}^{(k-1)} \right],$$

we define A_i for $i = (i_1, \dots, i_{2n})$ to be the set or indices with unique i 's, i.e the set containing those $l \in \{1, \dots, 2n\}$, s.t. $\#\{l' : i_l = i_{l'}\} = 1$. By Gaussian partial integration with respect to all g_{p, i_l} with $l \in A_i$ we obtain

$$\mathbb{E} \left[\prod_{l=1}^{2n} g_{p, i_l} \partial_\gamma m_{i_l}^{(k-1)} \right]^2 = \mathbb{E} \left[\left(\prod_{l \in A_i^c} g_{p, i_l} \right) \left(\prod_{l \in A_i} \partial_{p i_l} \right) \prod_{l=1}^{2n} \partial_\gamma m_{i_l}^{(k-1)} \right]^2,$$

which is by Cauchy-Schwarz at most

$$\leq \mathbb{E} \left[\left(\prod_{l \in A_i^c} g_{p, i_l} \right)^2 \right] \mathbb{E} \left[\left(\prod_{l \in A_i} \partial_{p i_l} \right) \prod_{l=1}^{2n} \partial_\gamma m_{i_l}^{(k-1)} \right]^2. \quad (55)$$

By Leibniz rule we have the identity

$$\left(\prod_{l \in A_i} \partial_{p i_l} \right) \prod_{l=1}^{2n} \partial_\gamma m_{i_l}^{(k-1)} = \sum_{\tau \in \mathcal{M}(i)} \prod_{l=1}^{2n} \partial_{\gamma + \tau_l} m_{i_l}^{(k-1)},$$

where

$$\mathcal{M}(i) := \left\{ \tau \in (\mathbb{N}_0^p)^{2n} : \sum_{l=1}^{2n} \tau_l = \sum_{l \in A_i} \delta_{\{p, i_l\}} \right\}.$$

Since the first term in (55) is $\preceq 1$ we see that (55) is bounded by

$$\preceq \sum_{\tau, \tau' \in \mathcal{M}(i)} \mathbb{E} \left[\left(\prod_{l=1}^{2n} \partial_{\gamma + \tau_l} m_{i_l}^{(k-1)} \right) \left(\prod_{l'=1}^{2n} \partial_{\gamma + \tau'_{l'}} m_{i_{l'}}^{(k-1)} \right) \right],$$

which in turn, estimating the product by the Hölder inequality (51) and using $|A_i| \preceq 1$, is bounded by

$$\preceq \left(\max_{\tau \in \mathcal{M}(i)} \prod_{l=1}^{2n} \mathbb{E} \left[\left(\partial_{\gamma + \tau_l} m_{i_l}^{(k-1)} \right)^{4n} \right]^{1/4n} \right)^2.$$

By the assertion of Theorem 9 for $k-1$,

$$\mathbb{E} \left[\left(\partial_{\gamma+\tau_l} m_i^{(k-1)} \right)^{4n} \right]^{1/4n} \preceq \sqrt{N}^{-|\gamma+\tau_l|-\mathcal{C}_{\gamma+\tau_l}(i_l)} = \sqrt{N}^{-|\gamma|-|\tau_l|-\mathcal{C}_{\gamma+\tau_l}(i_l)}.$$

Hence we overall obtain the estimate

$$\mathbb{E} \left[\left(\sum_{i=1}^N g_{p,i} \partial_{\gamma} m_i^{(k-1)} \right)^{2n} \right] \preceq \sum_{i_1, \dots, i_{2n}=1}^N N^{-n|\gamma|-|A_i|/2-\min_{\tau \in \mathcal{M}(j)} \sum_{l=1}^{2n} \mathcal{C}_{\gamma+\tau_l}(i_l)/2}. \quad (56)$$

Since $\mathcal{C}_{\gamma+\tau_l}(i_l)$ and $\mathcal{C}_{\gamma+\tau_l}(p)$ are identical up to one component that is possibly not counted, and since all edges in τ_l contain p , we have

$$\mathcal{C}_{\gamma+\tau_l}(i_l) \geq \mathcal{C}_{\gamma+\tau_l}(p) - \mathbf{1}_{i_l \in \gamma, \tau_l=0} \geq \mathcal{C}_{\gamma}(p) - \mathbf{1}_{i_l \in \gamma} - \sum_{l' \in A_i} \mathbf{1}_{i_{l'} \in \gamma, i_{l'} \in \tau_l}, \quad (57)$$

where the second estimate is due to each entry in τ_l at most removing a single component from the count by connecting it to p . By definition of A_i and τ note that $\sum_{l=1}^{2n} \mathbf{1}_{i_{l'} \in \tau_l} = 1$ for each $l' \in A_i$. Therefore summing (57) over l yields

$$\sum_{l=1}^{2n} \mathcal{C}_{\gamma+\tau_l}(i_l) \geq 2n\mathcal{C}_{\gamma}(p) - 2 \sum_{l \in A_i} \mathbf{1}_{i_l \in \gamma} - \sum_{l \in A_i^c} \mathbf{1}_{i_l \in \gamma}. \quad (58)$$

Now estimating the minimum in (56) with (58) yields

$$\mathbb{E} \left[\left(\sum_{i=1}^N g_{p,i} \partial_{\gamma} m_i^{(k-1)} \right)^{2n} \right] \preceq N^{n(1-|\gamma|-\mathcal{C}_{\gamma}(p))} \sum_{i_1, \dots, i_{2n}=1}^N N^{-n-|A_i|/2+\sum_{l \in A_i} \mathbf{1}_{i_l \in \gamma} + \sum_{l \in A_i^c} \mathbf{1}_{i_l \in \gamma}/2}$$

It remains to show, that the sum on the r.h.s. is $\preceq 1$ for any choice of γ with $|\gamma| \preceq 1$. To this end we decompose the sum into two parts. First we sum over all partitions of $\{1, \dots, 2n\}$, $P \in \mathcal{P}_{2n}$ say, and then we decide one by one for each block a number in $\{1, \dots, N\}$ that was not taken by a previous block. Here we decide first which of the i take the same values, and then we assign these values. This yields

$$\sum_{i_1, \dots, i_{2n}=1}^N N^{-n-|A_i|/2+\sum_{l \in A_i} \mathbf{1}_{i_l \in \gamma} + \sum_{l \in A_i^c} \mathbf{1}_{i_l \in \gamma}/2} = \sum_{P \in \mathcal{P}_{2n}} \sum_{\substack{1 \leq i_{B_1} < \dots < i_{B_L} \leq N \\ \{B_1, \dots, B_L\} = P}} \prod_{l=1}^L \sqrt{N}^{(|B_l| + \mathbf{1}_{|B_l|=1})(\mathbf{1}_{i_{B_l} \in \gamma} - 1)} \quad (59)$$

since

$$\begin{aligned} -2n - |A_i| + 2 \sum_{l \in A_i} \mathbf{1}_{i_l \in \gamma} + \sum_{l \in A_i^c} \mathbf{1}_{i_l \in \gamma} &= -2n - |A_i| + \sum_{l \in A_i} \mathbf{1}_{i_l \in \gamma} + \sum_{l=1}^{2n} \mathbf{1}_{i_l \in \gamma} \\ &= \sum_{l=1}^L (|B_l| + \mathbf{1}_{|B_l|=1})(\mathbf{1}_{i_{B_l} \in \gamma} - 1). \end{aligned}$$

Estimating by dropping the restriction to distinct values for i_B we obtain

$$(59) \leq \sum_{P \in \mathcal{P}_{2n}} \prod_{B \in P} \sum_{i_B=1}^N \sqrt{N}^{(|B| + \mathbf{1}_{|B|=1})(\mathbf{1}_{i_B \in \gamma} - 1)}.$$

If $i_B \in \gamma$, then the the exponent is zero and there are $|\gamma|$ such choices in the inner sum. If however $i_B \notin \gamma$, then the exponent is at most -2 and therefore the inner sum is at most

one. Hence the inner sum is at most $1 + |\gamma|$ and using that the number of partitions, factors and $|\gamma|$ are all ≤ 1 the claim follows. \square

Proof of Lemma 13. Expanding and using the Hölder inequality (51) yields

$$\begin{aligned} \mathbb{E} \left[\left(\sum_i \mathbf{1}_{\gamma_{\{p,i\}} > 0} \partial_{\gamma - \delta_{\{p,i\}}} m_i^{(k-1)} \right)^{2n} \right] &= \sum_{i_1, \dots, i_{2n}} \mathbb{E} \left[\prod_{l=1}^{2n} \mathbf{1}_{\gamma_{\{p,i_l\}} > 0} \partial_{\gamma - \delta_{\{p,i_l\}}} m_{i_l}^{(k-1)} \right] \\ &\leq \sum_{i_1, \dots, i_{2n}} \prod_{l=1}^{2n} \mathbf{1}_{\gamma_{\{p,i_l\}} > 0} \mathbb{E} \left[\left(\partial_{\gamma - \delta_{\{p,i_l\}}} m_{i_l}^{(k-1)} \right)^{2n} \right]^{1/2n} \\ &= \left(\sum_i \mathbf{1}_{\gamma_{\{p,i\}} > 0} \mathbb{E} \left[\left(\partial_{\gamma - \delta_{\{p,i\}}} m_i^{(k-1)} \right)^{2n} \right]^{1/2n} \right)^{2n}. \end{aligned}$$

By the assertion of Theorem 9 for $k-1$ we estimate

$$\leq \left(\sum_i \mathbf{1}_{\gamma_{\{p,i\}} > 0} (\sqrt{N})^{-|\gamma|+1-\mathcal{C}_{\gamma-\delta_{\{p,i\}}}(i)} \right)^{2n}.$$

The claim follows, since $|\{i : \gamma_{\{p,i\}} > 0\}| \leq 1$ and $\mathcal{C}_{\gamma-\delta_{\{p,i\}}}(i) \geq \mathcal{C}_{\gamma}(i) = \mathcal{C}_{\gamma}(p)$ if $\gamma_{\{p,i\}} > 0$. \square

Proof of Lemma 14. We start by assuming choice (II). Then, expanding and applying the Hölder inequality (51) inequality yields

$$\mathbb{E} \left[\left(\partial_{\gamma} \text{On}_p^{(k-1)} \right)^{2n} \right] \leq \left(\frac{\beta}{\sqrt{N}} \right)^{2n} \sum_{i_1, \dots, i_{2n}} \prod_{l=1}^{2n} \mathbb{E} \left[\left(\partial_{\gamma + \delta_{\{i_l, p\}}} m_{i_l}^{(k-1)} \right)^{2n} \right]^{1/2n}.$$

By the assertion of Theorem 9 for $k-1$ we further estimate

$$\leq \left(\frac{\beta}{\sqrt{N}} \right)^{2n} \sum_{i_1, \dots, i_{2n}} \prod_{l=1}^{2n} N^{-\frac{|\gamma|+1}{2} - \frac{\mathcal{C}_{\gamma+\delta_{\{i_l, p\}}}(i_l)}{2}} = \beta^{2n} N^{-2n-n|\gamma|} \sum_{i_1, \dots, i_{2n}} N^{-\frac{1}{2} \sum_{l=1}^{2n} \mathcal{C}_{\gamma+\delta_{\{i_l, p\}}}(i_l)}. \quad (60)$$

Using that that

$$\mathcal{C}_{\gamma+\delta_{\{i_l, p\}}}(i_l) = \mathcal{C}_{\gamma+\delta_{\{i_l, p\}}}(p) \geq \mathcal{C}_{\gamma}(p) - \mathbf{1}_{i_l \in \gamma},$$

we get

$$(60) \leq N^{-2n-n|\gamma|-n\mathcal{C}_{\gamma}(p)} \sum_{i_1, \dots, i_{2n}} N^{\frac{1}{2} \sum_{l=1}^{2n} \mathbf{1}_{i_l \in \gamma}} = N^{-n|\gamma|-n\mathcal{C}_{\gamma}(p)} \prod_{l=1}^{2n} \frac{1}{N} \sum_{i_l=1}^N \sqrt{N}^{\mathbf{1}_{i_l \in \gamma}}, \quad (61)$$

and the claim follows. Next, we consider choice (I). Using that $(a+b)^{2n} \leq 2^{2n} a^{2n} + 2^{2n} b^{2n}$, we obtain

$$\mathbb{E} \left[\left(\partial_{\gamma} \text{On}_p^{(k-1)} \right)^{2n} \right] \leq \mathbb{E} \left[\left(\partial_{\gamma} m_p^{(k-2)} \right)^{2n} \right] + N^{-2n} \mathbb{E} \left[\left(\sum_{i=1}^N \partial_{\gamma} \left(m_i^{(k-1)^2} m_p^{(k-2)} \right) \right)^{2n} \right]. \quad (62)$$

By the assertion of Theorem 9 we immediately obtain the desired $\preceq N^{-n|\gamma|-n\mathcal{C}_\gamma(p)}$ for the first summand. Analysing the second we observe by the Leibniz rule,

$$\partial_\gamma \left(m_i^{(k-1)^2} m_p^{(k-2)} \right) = \sum_{\substack{\rho, \sigma, \tau \in \mathbb{N}_0^p \\ \rho + \sigma + \tau = \gamma}} \binom{\gamma}{\rho \ \sigma \ \tau} \left(\partial_\rho m_i^{(k-1)} \right) \left(\partial_\sigma m_i^{(k-1)} \right) \left(\partial_\tau m_p^{(k-2)} \right).$$

Hence expanding and estimating by the Hölder inequality (51) yields

$$\begin{aligned} N^{-2n} \mathbb{E} \left[\left(\sum_{i=1}^N \partial_\gamma \left(m_i^{(k-1)^2} m_p^{(k-2)} \right) \right)^{2n} \right] \\ \preceq N^{-2n} \sum_{i_1, \dots, i_{2n}} \sum_{\substack{\rho_l, \sigma_l, \tau_l \in \mathbb{N}_0^p \\ \rho_l + \sigma_l + \tau_l = \gamma, l=1, \dots, 2n}} \prod_{l=1}^{2n} \mathbb{E} \left[\left(\partial_{\rho_l} m_{i_l}^{(k-1)} \right)^{6n} \right]^{1/6n} \\ \mathbb{E} \left[\left(\partial_{\sigma_l} m_{i_l}^{(k-1)} \right)^{6n} \right]^{1/6n} \mathbb{E} \left[\left(\partial_{\tau_l} m_p^{(k-2)} \right)^{6n} \right]^{1/6n}. \end{aligned}$$

By the assertion of Theorem 9 we further estimate

$$\preceq N^{-2n} \sum_{i_1, \dots, i_{2n}} \sum_{\substack{\rho_l, \sigma_l, \tau_l \in \mathbb{N}_0^p \\ \rho_l + \sigma_l + \tau_l = \gamma, l=1, \dots, 2n}} \prod_{l=1}^{2n} N^{-\frac{|\rho_l|}{2} - \frac{\mathcal{C}_{\rho_l}(i_l)}{2}} N^{-\frac{|\sigma_l|}{2} - \frac{\mathcal{C}_{\sigma_l}(i_l)}{2}} N^{-\frac{|\tau_l|}{2} - \frac{\mathcal{C}_{\tau_l}(p)}{2}}. \quad (63)$$

By definition of \mathcal{C}

$$\mathcal{C}_{\rho_l}(i_l) \geq \mathcal{C}_{\rho_l}(p) - \mathbf{1}_{i_l \in \gamma}, \quad \mathcal{C}_{\sigma_l}(i_l) \geq \mathcal{C}_{\sigma_l}(p) - \mathbf{1}_{i_l \in \gamma}$$

as well as

$$\mathcal{C}_{\rho_l}(p) + \mathcal{C}_{\sigma_l}(p) + \mathcal{C}_{\tau_l}(p) \geq \mathcal{C}_{\rho_l + \sigma_l + \tau_l}(p) = \mathcal{C}_\gamma(p)$$

hold. Using these facts as well as $|\gamma| \preceq 1$ the claim follows:

$$(63) \preceq N^{-n|\gamma|-n\mathcal{C}_\gamma(p)} \sum_{i_1, \dots, i_{2n}} N^{\sum_{l=1}^{2n} (\mathbf{1}_{i_l \in \gamma} - 1)} = N^{-n|\gamma|-n\mathcal{C}_\gamma(p)} \prod_{l=1}^{2n} \sum_{i=1}^N N^{\mathbf{1}_{i \in \gamma} - 1} \preceq N^{-n|\gamma|-n\mathcal{C}_\gamma(p)} \quad \square$$

4.3. Estimating error terms / Proof of Theorem 8. Theorem 8 is a direct consequence of the following Lemmata by considering $\alpha = 0, t = 1$ in Lemma 15 and $\alpha = \delta_{\{y,z\}}, t = 1$ as well as $\alpha = 0, t = 1$ in Lemma 16. We write \mathcal{C}_α for the number of connected components in the multigraph associated to α .

Lemma 15. For $\beta, h \in \mathbb{R}, k, t \in \mathbb{N}, N \in \mathbb{N}, x, y, z \leq N$ with $y \neq z, \alpha \in \mathbb{N}_0^p$, there exists $C = C_{k, |\alpha|, \beta, h, t} > 0$, s.t.

$$\mathbb{E} \left[\left(\partial_\alpha \left(\Delta_{x; y, z}^{(k)} - \partial_{\delta_{\{y, z\}}} \mathcal{E}_x^{(k)} \right) \right)^{2t} \right] \leq CN^{-t} (|\alpha| + \mathbf{1}_{x \neq \alpha} + \mathcal{C}_{\alpha + \delta_{\{y, z\}}}).$$

Lemma 16. For $\beta \geq 0, h \in \mathbb{R}, k, t \in \mathbb{N}, N \in \mathbb{N}, x \leq N, \alpha \in \mathbb{N}_0^p$, there exists $C = C_{k, |\alpha|, \beta, h, t} > 0$, s.t.

$$\mathbb{E} \left[\left(\partial_\alpha \mathcal{E}_x^{(k)} \right)^{2t} \right] \leq CN^{-t} (|\alpha| + \mathbf{1}_{x \neq \alpha} + \mathcal{C}_\alpha).$$

Proof of Lemma 15. Writing $\hat{\alpha} := \alpha + \delta_{\{y,z\}}$ to shorten notation we have by definition of Δ and \mathcal{E} that

$$\partial_{\alpha} \left(\Delta_{x;y,z}^{(k)} - \partial_{\delta_{\{y,z\}}} \mathcal{E}_x^{(k)} \right) = \frac{\beta}{\sqrt{N}} \sum_j \left(g_{j,x} \partial_{\hat{\alpha}} m_j^{(k)} - \partial_{\hat{\alpha} + \delta_{\{j,x\}}} m_j^{(k)} \right) + \frac{\beta}{\sqrt{N}} \sum_j \mathbf{1}_{\{j,x\} \in \alpha} \partial_{\hat{\alpha} - \delta_{\{j,x\}}} m_j^{(k)}.$$

Hence

$$\begin{aligned} & \mathbb{E} \left[\left(\partial_{\alpha} \left(\Delta_{x;y,z}^{(k)} - \partial_{\delta_{\{y,z\}}} \mathcal{E}_x^{(k)} \right) \right)^{2t} \right] \\ & \preceq N^{-t} \mathbb{E} \left[\left(\sum_j \left(g_{j,x} \partial_{\hat{\alpha}} m_j^{(k)} - \partial_{\hat{\alpha} + \delta_{\{j,x\}}} m_j^{(k)} \right) \right)^{2t} \right] + N^{-t} \mathbb{E} \left[\left(\sum_j \mathbf{1}_{\{j,x\} \in \alpha} \partial_{\hat{\alpha} - \delta_{\{j,x\}}} m_j^{(k)} \right)^{2t} \right]. \end{aligned} \quad (64)$$

Using that $|\alpha|, t \preceq 1$ we can estimate the second expectation by

$$\mathbb{E} \left[\left(\sum_j \mathbf{1}_{\{j,x\} \in \alpha} \partial_{\hat{\alpha} - \delta_{\{j,x\}}} m_j^{(k)} \right)^{2t} \right] \preceq \sum_j \mathbf{1}_{\{j,x\} \in \alpha} \mathbb{E} \left[\left(\partial_{\hat{\alpha} - \delta_{\{j,x\}}} m_j^{(k)} \right)^{2t} \right] \preceq N^{-t} (|\alpha| + \mathcal{C}_{\hat{\alpha}} - \mathbf{1}_{x \in \alpha}) \quad (65)$$

the last step by Theorem 9 and since for $x \in \alpha$ we have $\mathcal{C}_{\hat{\alpha} - \delta_{\{j,x\}}}(j) \geq \mathcal{C}_{\hat{\alpha}}(x) \geq \mathcal{C}_{\hat{\alpha}} - \mathbf{1}_{x \in \alpha}$. It remains to control the first expectation in (64). To this end we expand and collect same j , i.e. for \mathcal{P}_{2t} the set of partitions of $\{1, \dots, 2t\}$ and writing $[i]$ for the block in P containing i we have

$$\begin{aligned} & \mathbb{E} \left[\left(\sum_j \left(g_{j,x} \partial_{\hat{\alpha}} m_j^{(k)} - \partial_{\hat{\alpha} + \delta_{\{j,x\}}} m_j^{(k)} \right) \right)^{2t} \right] = \\ & \sum_{P \in \mathcal{P}_{2t}} \sum_{\substack{1 \leq j_{B_1} < \dots < j_{B_L} \leq N \\ \{B_1, \dots, B_L\} = P}} \mathbb{E} \left[\prod_{i=1}^{2t} \left(g_{j_{[i]},x} \partial_{\hat{\alpha}} m_{j_{[i]}}^{(k)} - \partial_{\hat{\alpha} + \delta_{\{j_{[i]},x\}}} m_{j_{[i]}}^{(k)} \right) \right] \end{aligned}$$

By Gaussian partial integration a factor $g_{j_{[i]},x} \partial_{\gamma} m_{j_{[i]}}^{(k)} - \partial_{\gamma + \delta_{\{j_{[i]},x\}}} m_{j_{[i]}}^{(k)}$ has the same effect as taking the derivative w.r.t. $g_{j_{[i]},x}$ and then multiplying by $\partial_{\gamma} m_{j_{[i]}}^{(k)}$. I.e. for any random variable Z such that the following is defined we have $\mathbb{E} \left[\left(g_{j_{[i]},x} \partial_{\gamma} m_{j_{[i]}}^{(k)} - \partial_{\gamma + \delta_{\{j_{[i]},x\}}} m_{j_{[i]}}^{(k)} \right) Z \right] = \mathbb{E} \left[\partial_{\gamma} m_{j_{[i]}}^{(k)} \partial_{\delta_{\{j_{[i]},x\}}} Z \right]$. Applying this for all i such that $|[i]| = 1$ and by product rule we obtain the representation

$$= \sum_{P \in \mathcal{P}_{2t}} \sum_{\substack{1 \leq j_{B_1} < \dots < j_{B_L} \leq N \\ \{B_1, \dots, B_L\} = P}} \sum_{\tau} \mathbb{E} \left[\prod_{\substack{i=1, \dots, 2t: \\ |[i]|=1}} \partial_{\hat{\alpha} + \tau_i} m_{j_{[i]}}^{(k)} \prod_{\substack{i=1, \dots, 2t: \\ |[i]|>1}} \left(g_{j_{[i]},x} \partial_{\hat{\alpha} + \tau_i} m_{j_{[i]}}^{(k)} - \partial_{\hat{\alpha} + \tau_i + \delta_{\{j_{[i]},x\}}} m_{j_{[i]}}^{(k)} \right) \right],$$

where the τ -sum is over all $\tau_1, \dots, \tau_{2t} \in \mathbb{N}_0^p$, s.t. $\sum_i \tau_i = \sum_{i:|[i]|=1} \delta_{\{j_{[i]}, x\}}$ and $\{j_{[i]}, x\} \notin \tau_i$. Hence estimating by generalized Hölder inequality yields

$$\begin{aligned} &\preceq \sum_{P \in \mathcal{P}_{2t}} \sum_{\substack{1 \leq j_{B_1} < \dots < j_{B_L} \leq N \\ \{B_1, \dots, B_L\} = P}} \sum_{\tau} \prod_{i:|[i]|=1} \mathbb{E} \left[\left(\partial_{\hat{\alpha} + \tau_i} m_{j_{[i]}}^{(k)} \right)^{2t} \right]^{1/2t} \\ &\quad \prod_{i:|[i]|>1} \mathbb{E} \left[\left(g_{j_{[i]}, x} \partial_{\hat{\alpha} + \tau_i} m_{j_{[i]}}^{(k)} - \partial_{\hat{\alpha} + \tau_i + \delta_{\{j_{[i]}, x\}}} m_{j_{[i]}}^{(k)} \right)^{2t} \right]^{1/2t}. \end{aligned} \quad (66)$$

Using that $\mathbb{E}[g_{\cdot, \cdot}^{2t}] \preceq 1$ for any $t \preceq 1$, Hölder inequality and applying Theorem 9 gives

$$\begin{aligned} &\mathbb{E} \left[\left(g_{j_{[i]}, x} \partial_{\hat{\alpha} + \tau_i} m_{j_{[i]}}^{(k)} \right)^{2t} \right], \mathbb{E} \left[\left(\partial_{\hat{\alpha} + \tau_i + \delta_{\{j_{[i]}, x\}}} m_{j_{[i]}}^{(k)} \right)^{2t} \right], \mathbb{E} \left[\left(\partial_{\hat{\alpha} + \tau_i} m_{j_{[i]}}^{(k)} \right)^{2t} \right] \\ &\preceq N^{-t(|\hat{\alpha}| + |\tau_i| + \mathcal{C}_{\hat{\alpha} + \tau_i}(j_{[i]}))}. \end{aligned} \quad (67)$$

Hence we obtain estimating (66)

$$\begin{aligned} &\mathbb{E} \left[\left(\sum_j \left(g_{j, x} \partial_{\hat{\alpha}} m_j^{(k)} - \partial_{\hat{\alpha} + \delta_{\{j, x\}}} m_j^{(k)} \right) \right)^{2t} \right] \\ &\preceq \sum_{P \in \mathcal{P}_{2t}} \sum_{\substack{1 \leq j_{B_1} < \dots < j_{B_L} \leq N \\ \{B_1, \dots, B_L\} = P}} \sum_{\tau} \prod_{i=1}^{2t} N^{-\frac{1}{2}(|\hat{\alpha}| + |\tau_i| + \mathcal{C}_{\hat{\alpha} + \tau_i}(j_{[i]}))}. \end{aligned} \quad (68)$$

Estimating $\mathcal{C}_{\hat{\alpha} + \tau_i}(j_{[i]}) \geq \mathcal{C}_{\hat{\alpha}} - \mathbf{1}_{j_{[i]} \in \hat{\alpha}} - \sum_{i'} \mathbf{1}_{j_{[i']} \in \tau_i, j_{[i']} \in \hat{\alpha}}$ and collecting terms using $|[i']| = 1 \Leftrightarrow j_{[i']} \in \sum_i \tau_i$ we obtain

$$\preceq N^{-t(|\alpha| + \mathcal{C}_{\hat{\alpha}})} \sum_{P \in \mathcal{P}_{2t}} \sum_{\substack{1 \leq j_{B_1} < \dots < j_{B_L} \leq N \\ \{B_1, \dots, B_L\} = P}} \sum_{\tau} N^{\sum_i \left(\frac{1+|[i]|}{2} \right)} (\mathbf{1}_{j_{[i]} \in \hat{\alpha}} - 1).$$

Estimating further by dropping the requirement for distinct j we obtain the result

$$\preceq N^{-t(|\alpha| + \mathcal{C}_{\hat{\alpha}})} \sum_{P \in \mathcal{P}_{2t}} \prod_{b=1}^L \sum_{j=1}^N N^{\left(\frac{|B_b|+1}{2} \right)} (\mathbf{1}_{j \in \hat{\alpha}} - 1) \preceq N^{-t(|\alpha| + \mathcal{C}_{\hat{\alpha}})}. \quad (69)$$

The last step since the sums concerning P and τ only have $\preceq 1$ many summands and the innermost sum is bounded by $|\hat{\alpha}| + 1$ as the exponent of N is at most -1 except for the $|\hat{\alpha}|$ many cases where it is zero. Plugging (65) and (69) into (64) yields the claim. \square

Proof of Lemma 16. It suffices to consider choice (I) as the assertion is immediate for choice (II).

We recall that $m_l^{(k)} = \tanh(h + Y_l^{(k-1)})$. By the chain rule and (47), we have

$$\begin{aligned} \partial_{x,l} m_l^{(k)} &= \left(1 - m_l^{(k)2}\right) \partial_{x,l} Y_l^{(k-1)} \\ &= \left(1 - m_l^{(k)2}\right) \left[\frac{\beta}{\sqrt{N}} \sum_{p \neq l} \delta_{\{l,p\}\{x,l\}} m_p^{(k-1)} + \Delta_{l;x,l}^{(k-1)} \right] \\ &= \left(1 - m_l^{(k)2}\right) \left[\frac{\beta}{\sqrt{N}} m_x^{(k-1)} + \Delta_{l;x,l}^{(k-1)} \right]. \end{aligned} \quad (70)$$

Hence,

$$\mathcal{E}_x^{(k)} = \frac{\beta}{\sqrt{N}} \sum_l \partial_{x,l} m_l^{(k)} - \beta^2 \left(1 - q_N^{(k)}\right) m_x^{(k-1)} = \frac{\beta}{\sqrt{N}} \sum_l \left(1 - m_l^{(k)2}\right) \Delta_{l;x,l}^{(k-1)}.$$

By the product rule, it follows that

$$\begin{aligned} \partial_\alpha \mathcal{E}_x^{(k)} &= \frac{\beta}{\sqrt{N}} \sum_l \partial_\alpha \left(1 - m_l^{(k)}\right) \left(1 + m_l^{(k)}\right) \Delta_{l;x,l}^{(k-1)} \\ &= \frac{\beta}{\sqrt{N}} \sum_l \sum_{\substack{\gamma, \delta, \varepsilon \in \mathbb{N}_0^p: \\ \gamma + \delta + \varepsilon = \alpha}} \binom{\alpha}{\gamma \delta \varepsilon} \left(\mathbf{1}_{|\gamma|=0} - \partial_\gamma m_l^{(k)}\right) \left(\mathbf{1}_{|\delta|=0} + \partial_\delta m_l^{(k)}\right) \partial_\varepsilon \Delta_{l;x,l}^{(k-1)}. \end{aligned}$$

Hence, by expansion and the Hölder inequality (51),

$$\begin{aligned} &\mathbb{E} \left[\left(\partial_\alpha \mathcal{E}_x^{(k)} \right)^{2t} \right] \\ &\preceq N^{-t} \sum_{l_1, \dots, l_{2t}=1}^N \max_{\substack{\gamma_b, \delta_b, \varepsilon_b \in \mathbb{N}_0^p: \\ \gamma_b + \delta_b + \varepsilon_b = \alpha}} \mathbb{E} \left[\prod_{b=1}^{2t} \left(\mathbf{1}_{|\gamma_b|=0} - \partial_{\gamma_b} m_{l_b}^{(k)} \right) \left(\mathbf{1}_{|\delta_b|=0} + \partial_{\delta_b} m_{l_b}^{(k)} \right) \partial_{\varepsilon_b} \Delta_{l_b;x,l_b}^{(k-1)} \right] \\ &\leq N^{-t} \sum_{l_1, \dots, l_{2t}=1}^N \max_{\substack{\gamma_b, \delta_b, \varepsilon_b \in \mathbb{N}_0^p: \\ \gamma_b + \delta_b + \varepsilon_b = \alpha}} \prod_{b=1}^{2t} \mathbb{E} \left[\left(\mathbf{1}_{|\gamma_b|=0} - \partial_{\gamma_b} m_{l_b}^{(k)} \right)^{6t} \right]^{1/6t} \mathbb{E} \left[\left(\mathbf{1}_{|\delta_b|=0} + \partial_{\delta_b} m_{l_b}^{(k)} \right)^{6t} \right]^{1/6t} \\ &\quad \times \mathbb{E} \left[\left(\partial_{\varepsilon_b} \Delta_{l_b;x,l_b}^{(k-1)} \right)^{6t} \right]^{1/6t} \end{aligned} \quad (71)$$

As $(a+b)^{6t} \preceq a^{6t} + b^{6t}$ and by Theorem 9,

$$\mathbb{E} \left[\left(\partial_{\varepsilon_b} \Delta_{l_b;x,l_b}^{(k-1)} \right)^{6t} \right] \preceq \mathbb{E} \left[\left(\partial_{\varepsilon_b} \partial_{\delta_{\{x,l_b\}}} \mathcal{E}_{l_b}^{(k-1)} \right)^{6t} \right] + \mathbb{E} \left[\left(\partial_{\varepsilon_b} \left(\Delta_{l_b;x,l_b}^{(k-1)} - \partial_{\delta_{\{x,l_b\}}} \mathcal{E}_{l_b}^{(k-1)} \right) \right)^{6t} \right] \quad (72)$$

and

$$\mathbb{E} \left[\left(\mathbf{1}_{|\gamma_b|=0} - \partial_{\gamma_b} m_{l_b}^{(k)} \right)^{6t} \right] \preceq \mathbf{1}_{|\gamma_b|=0} + \mathbb{E} \left[\left(\partial_{\gamma_b} m_{l_b}^{(k)} \right)^{6t} \right] \preceq N^{-3t|\gamma_b| - 3t\mathcal{C}_{\gamma_b}(l_b)}.$$

The second term on the rhs of (72) is $\preceq N^{-3t \left(|\varepsilon_b| + \mathcal{C}_{\varepsilon_b + \delta_{\{x,l_b\}}} + \mathbf{1}_{l_b \neq \varepsilon_b} \right)}$ by Lemma 15. We conclude by induction. The assertion holds for $k = 1$ as $\mathcal{E}^{(1)} \equiv 0$, since $m^{(0)} = 0$ and $m^{(1)}$ is independent of g . From now on, we assume that the assertion holds for $k - 1$ for

some $k \geq 2$. Then, the first term on the rhs of (72) is $\leq N^{-3t} \left(|\varepsilon_b + \delta_{\{x, l_b\}}| + \mathcal{C}_{\varepsilon_b + \delta_{\{x, l_b\}}} \right)$ by the induction hypothesis.

Plugging these bounds into (71), and using that

$$|\gamma_b| + |\delta_b| + |\varepsilon_b| = |\alpha|,$$

$$\mathcal{C}_{\gamma_b}(l_b) \geq \mathcal{C}_{\gamma_b} - \mathbf{1}_{l_b \in \gamma_b}, \quad \mathcal{C}_{\gamma_b} + \mathcal{C}_{\delta_b} + \mathcal{C}_{\varepsilon_b + \delta_{\{x, l_b\}}} \geq \mathcal{C}_{\alpha + \delta_{\{x, l_b\}}} \geq \mathcal{C}_{\alpha} + \mathbf{1}_{x \notin \alpha} - \mathbf{1}_{l_b \in \alpha},$$

as well as $|\alpha| \leq 1$, we obtain

$$\begin{aligned} & \mathbb{E} \left[\left(\partial_{\alpha} \mathcal{E}_x^{(k)} \right)^{2t} \right] \\ & \leq N^{-t} \sum_{l_1, \dots, l_{2t}=1}^N \max_{\substack{\gamma_b, \delta_b, \varepsilon_b \in \mathbb{N}_0^p: \\ \gamma_b + \delta_b + \varepsilon_b = \alpha}} \prod_{b=1}^{2t} N^{-\frac{1}{2}} \left(|\gamma_b| + \mathcal{C}_{\gamma_b}(l_b) + |\delta_b| + \mathcal{C}_{\delta_b}(l_b) + |\varepsilon_b| + \mathcal{C}_{\varepsilon_b + \delta_{\{x, l_b\}}} + \mathbf{1}_{l_b \notin \varepsilon_b} \right) \\ & \leq N^{-t} (|\alpha| + \mathcal{C}_{\alpha} + \mathbf{1}_{x \notin \alpha})^{-2t} \sum_{l_1, \dots, l_{2t}=1}^N \max_{\substack{\gamma_b, \delta_b, \varepsilon_b \in \mathbb{N}_0^p: \\ \gamma_b + \delta_b + \varepsilon_b = \alpha}} \prod_{b=1}^{2t} N^{\frac{1}{2}} \left(\mathbf{1}_{l_b \in \gamma_b} + \mathbf{1}_{l_b \in \delta_b} + \mathbf{1}_{l_b \in \varepsilon_b} + \mathbf{1}_{l_b \in \alpha} \right) \\ & = N^{-t} (|\alpha| + \mathcal{C}_{\alpha} + \mathbf{1}_{x \notin \alpha})^{-2t} \left(\sum_{l=1}^N N^{\mathbf{1}_{l \in \alpha}} \right)^{2t} \leq N^{-t} (|\alpha| + \mathcal{C}_{\alpha} + \mathbf{1}_{x \notin \alpha}) \end{aligned}$$

as required. \square

5. COVARIANCES OF LOCAL FIELDS

The aim of this section is to prove the propagation relation from Theorem 7. As first step, we use Theorem 6 to show the propagation relation in expectation:

Lemma 17. *For $\beta, h \in \mathbb{R}$, $k, k' \geq 2$, there exists $C \equiv C_{k, k', \beta, h} > 0$ such that*

$$\left| \mathbb{E} \left[q_N^{(k, k')} \right] - \mathbb{E} \left[\psi \left(q_N^{(k-1, k'-1)}, q_N^{(k-1)}, q_N^{(k'-1)} \right) \right] \right| \leq \frac{C}{\sqrt{N}}$$

and

$$\left| \mathbb{E} \left[q_N^{(k, 1)} \right] - \mathbb{E} \left[\chi \left(q_N^{(k-1)} \right) \right] \frac{1}{N} \sum_{i=1}^N m_i^{(1)} \right| \leq \frac{C}{\sqrt{N}}.$$

Proof. By the definition (12) of $q_N^{(k, k')}$ and the TAP recursion (9),

$$\mathbb{E} \left[q_N^{(k, k')} \right] = \frac{1}{N} \sum_{i=1}^N \mathbb{E} \left[m_i^{(k)} m_i^{(k')} \right] = \frac{1}{N} \sum_{i=1}^N \mathbb{E} \left[\tanh \left(h + Y_i^{(k-1)} \right) \tanh \left(h + Y_i^{(k'-1)} \right) \right].$$

By Theorem 6,

$$= \frac{1}{N} \mathbb{E} \left[\psi \left(q_N^{(k-1, k'-1)}, q_N^{(k-1)}, q_N^{(k'-1)} \right) \right] + O(N^{-1/2}).$$

Analogously,

$$\begin{aligned}\mathbb{E}\left[q_N^{(k,1)}\right] &= \frac{1}{N} \sum_{i=1}^N \mathbb{E}\left[m_i^{(k)}\right] m_i^{(1)} = \frac{1}{N} \sum_{i=1}^N \mathbb{E}\left[\tanh\left(h + Y_i^{(k-1)}\right)\right] m_i^{(1)} \\ &= \mathbb{E}\left[\chi\left(q_N^{(k-1)}\right)\right] \frac{1}{N} \sum_{i=1}^N m_i^{(1)} + O(N^{-1/2}).\end{aligned}$$

□

Next we show, using Theorem 6 again, that $q_N^{(k,k')}$ concentrates around its expectation:

Lemma 18. *For $\beta, h \in \mathbb{R}$, $k, k' \in \mathbb{N}$, there exists $C \equiv C_{k,k',\beta,h} > 0$ such that*

$$\mathbb{E}\left[\left(q_N^{(k,k')} - \mathbb{E}\left[q_N^{(k,k')}\right]\right)^2\right] \leq \frac{C}{\sqrt{N}}$$

for all $N \in \mathbb{N}$.

Proof. We first consider the case that $k, k' \geq 2$. By the TAP recursion (9),

$$\mathbb{E}\left[q_N^{(k,k')}\right] = \sum_{i=1}^N \frac{1}{N} \mathbb{E}\left[m_i^{(k)} m_i^{(k')}\right] = \sum_{i=1}^N \frac{1}{N} \mathbb{E}\left[\tanh\left(h + Y_i^{(k-1)}\right) \tanh\left(h + Y_i^{(k'-1)}\right)\right].$$

With $f(x, y) = \tanh(h+x) \tanh(h+y)$ and \mathbf{Z} defined in Theorem 6, we obtain

$$= \sum_{i=1}^N \frac{1}{N} \mathbb{E}\left[\tanh\left(h + Z_i^{(k-1)}\right) \tanh\left(h + Z_i^{(k'-1)}\right)\right] + O(1/\sqrt{N}).$$

For the second moment, we obtain

$$\begin{aligned}\mathbb{E}\left[\left(q_N^{(k,k')}\right)^2\right] &= \frac{1}{N^2} \sum_{i,j=1}^N \mathbb{E}\left[m_i^{(k)} m_i^{(k')} m_j^{(k)} m_j^{(k')}\right] \\ &= \frac{1}{N^2} \sum_{i,j=1}^N \mathbb{E}\left[\tanh\left(h + Y_i^{(k-1)}\right) \tanh\left(h + Y_i^{(k'-1)}\right) \tanh\left(h + Y_j^{(k-1)}\right) \tanh\left(h + Y_j^{(k'-1)}\right)\right].\end{aligned}$$

Again by Theorem 6 with $f(x, y, z, w) = \tanh(h+x) \tanh(h+y) \tanh(h+z) \tanh(h+w)$,

$$\begin{aligned}&= \frac{1}{N^2} \sum_{i,j=1}^N \mathbb{E}\left[\tanh\left(h + Z_i^{(k-1)}\right) \tanh\left(h + Z_i^{(k'-1)}\right)\right. \\ &\quad \left. \times \tanh\left(h + Z_j^{(k-1)}\right) \tanh\left(h + Z_j^{(k'-1)}\right)\right] + O(1/\sqrt{N}).\end{aligned}$$

By the independence structure of the covariance matrix Σ ,

$$\begin{aligned}&= \frac{1}{N^2} \sum_{i,j=1}^N \mathbb{E}\left[\tanh\left(h + Z_i^{(k-1,i)}\right) \tanh\left(h + Z_i^{(k'-1)}\right)\right] \\ &\quad \times \mathbb{E}\left[\tanh\left(h + Z_j^{(k-1)}\right) \tanh\left(h + Z_j^{(k'-1)}\right)\right] + O(1/\sqrt{N}) \\ &= \mathbb{E}\left[\frac{1}{N} m^{(k)T} m^{(k')}\right]^2 + O(1/\sqrt{N})\end{aligned}$$

Hence,

$$\begin{aligned} & \mathbb{E} \left[\left(\frac{1}{N} m^{(k)T} m^{(k')} - \mathbb{E} \left[\frac{1}{N} m^{(k)T} m^{(k')} \right] \right)^2 \right] \\ &= \mathbb{E} \left[\left(\frac{1}{N} m^{(k)T} m^{(k')} \right)^2 \right] - \mathbb{E} \left[\frac{1}{N} m^{(k)T} m^{(k')} \right]^2 = O(1/\sqrt{N}). \end{aligned}$$

Next we consider the case that $k \geq 2$, $k' = 1$. Using that the initial value $m^{(1)}$ is independent of g , we obtain

$$\mathbb{E} \left[m_i^{(k)} m_i^{(1)} \right] = \mathbb{E} \left[m_i^{(k)} \right] m_i^{(1)} = \mathbb{E} \left[\tanh \left(h + Z_i^{(k-1)} \right) \right] m_i^{(1)} + O(1/\sqrt{N})$$

and

$$\begin{aligned} \mathbb{E} \left[m_i^{(k)} m_i^{(1)} m_j^{(k)} m_j^{(1)} \right] &= \mathbb{E} \left[m_i^{(k)} m_j^{(k)} \right] m_i^{(1)} m_j^{(1)} \\ &= \mathbb{E} \left[\tanh \left(h + Z_i^{(k-1)} \right) \tanh \left(h + Z_j^{(k-1)} \right) \right] m_i^{(1)} m_j^{(1)} + O(1/\sqrt{N}) \end{aligned}$$

and the above computation for $k, k' \geq 2$ passes through in this case just the same.

The case $k = k' = 1$ is trivial as $m^{(1)}$ is assumed independent of g . \square

We also show concentration of $\psi(q_N^{(k,k')}, q_N^{(k)}, q_N^{(k')})$ and of $\chi(q_N^{(k)})$ around the expectation. To this end, we establish the following Lemma:

Lemma 19. *For (X, Y) centred multivariate normal distributed and $f, g \in \mathcal{C}^2$, bounded and with two bounded derivatives we have*

$$\begin{aligned} \frac{d}{d \text{Cov}(X, Y)} \mathbb{E}(f(X) g(Y)) &= \mathbb{E}(f'(X) g'(Y)) \\ \frac{d}{d \text{Var}(X)} \mathbb{E}(f(X) g(Y)) &= \frac{1}{2} \mathbb{E}(f''(X) g(Y)) \\ \frac{d}{d \text{Var}(Y)} \mathbb{E}(f(X) g(Y)) &= \frac{1}{2} \mathbb{E}(f(X) g''(Y)) \end{aligned}$$

Remark 20. *Since we are interested in tanh and its derivatives for f and g the boundedness assumptions will always be satisfied for the purposes of this paper. They allow us to not worry about existence of expectations, but can be weakened significantly.*

Proof. We consider the covariance matrix

$$\mathcal{T} = \begin{pmatrix} t_2 & t_1 \\ t_1 & t_3 \end{pmatrix},$$

with $t_2, t_3 > 0$ and $|t_1| < \sqrt{t_2 t_3}$, i.e. we restrict ourselves to the interior of the set of covariance matrices. The general case follows by continuity. For a pair $Z = (Z_1, Z_2)$ of independent standard Gaussians, we construct $(X, Y) := Z^T \mathcal{T}^{1/2}$. Let σ_1, σ_2 denote the column vectors of $\mathcal{T}^{1/2}$. We note that restriction to the interior of the set of covariance matrices guarantees existence of derivatives of $\mathcal{T}^{1/2}$ with respect to t_1, t_2, t_3 , which will be needed in a moment. For now we obtain that

$$\mathbb{E}(f(X) g(Y)) = \mathbb{E} \left[f(\sigma_1^T Z) g(\sigma_2^T Z) \right].$$

Using Gaussian integration by parts and the shorthand $\partial_{t_i} = \frac{d}{dt_i}$ one quickly checks the following preparatory calculations

$$\begin{aligned}\mathbb{E}(Z_1(\partial_{t_i}\sigma_{1,1})f'(X)g(Y)) &= \mathbb{E}((\partial_{t_i}\sigma_{1,1})\sigma_{1,1}f''(X)g(Y)) + \mathbb{E}((\partial_{t_i}\sigma_{1,1})\sigma_{2,1}f'(X)g'(Y)), \\ \mathbb{E}(Z_2(\partial_{t_i}\sigma_{1,2})f'(X)g(Y)) &= \mathbb{E}((\partial_{t_i}\sigma_{1,2})\sigma_{1,2}f''(X)g(Y)) + \mathbb{E}((\partial_{t_i}\sigma_{1,2})\sigma_{2,2}f'(X)g'(Y)), \\ \mathbb{E}(Z_1(\partial_{t_i}\sigma_{2,1})f(X)g'(Y)) &= \mathbb{E}((\partial_{t_i}\sigma_{2,1})\sigma_{1,1}f'(X)g'(Y)) + \mathbb{E}((\partial_{t_i}\sigma_{2,1})\sigma_{2,1}f(X)g''(Y)), \\ \mathbb{E}(Z_2(\partial_{t_i}\sigma_{2,2})f(X)g'(Y)) &= \mathbb{E}((\partial_{t_i}\sigma_{2,2})\sigma_{1,2}f'(X)g'(Y)) + \mathbb{E}((\partial_{t_i}\sigma_{2,2})\sigma_{2,2}f(X)g''(Y)).\end{aligned}$$

We now compute directly

$$\begin{aligned}\partial_{t_i}\mathbb{E}(f(X)g(Y)) &= \partial_{t_i}\mathbb{E}_Z[f(\sigma_1^T Z)g(\sigma_2^T Z)] \\ &= \mathbb{E}(Z_1(\partial_{t_i}\sigma_{1,1})f'(X)g(Y)) + \mathbb{E}(Z_2(\partial_{t_i}\sigma_{1,2})f'(X)g(Y)) + \\ &\quad + \mathbb{E}(Z_1(\partial_{t_i}\sigma_{2,1})f(X)g'(Y)) + \mathbb{E}(Z_2(\partial_{t_i}\sigma_{2,2})f(X)g'(Y))\end{aligned}$$

and by our preparation

$$\begin{aligned}&= \frac{1}{2}\mathbb{E}((\partial_{t_i}(\sigma_{1,1}^2 + \sigma_{1,2}^2))f''(X)g(Y)) + \frac{1}{2}\mathbb{E}((\partial_{t_i}(\sigma_{2,2}^2 + \sigma_{2,1}^2))f(X)g''(Y)) + \\ &\quad + \mathbb{E}((\partial_{t_i}(\sigma_{1,1}\sigma_{2,1} + \sigma_{1,2}\sigma_{2,2}))f'(X)g'(Y)),\end{aligned}$$

which simply is

$$= \frac{1}{2}\mathbb{E}((\partial_{t_i}t_2)f''(X)g(Y)) + \frac{1}{2}\mathbb{E}((\partial_{t_i}t_3)f(X)g''(Y)) + \mathbb{E}((\partial_{t_i}t_1)f'(X)g'(Y)),$$

since $\mathcal{T}^{1/2}$ is symmetric. The claim follows by choosing $i = 1, 2, 3$. \square

Corollary 21. *The functions $\psi : [-1, 1] \times [0, 1]^2 \rightarrow [-1, 1]$ and $\chi : [0, 1] \rightarrow [-1, 1]$ (and their partial derivatives) are Lipschitz.*

Proof. It is a direct consequence of Lemma 19 since all derivatives of \tanh are bounded. \square

Corollary 22. *For $\beta, h \in \mathbb{R}$, $k, k' \in \mathbb{N}$, there exists $C \equiv C_{k, k', \beta, h} > 0$ such that*

$$\mathbb{E}\left[\left(\psi\left(q_N^{(k, k')}, q_N^{(k)}, q_N^{(k')}\right) - \mathbb{E}\left[\psi\left(q_N^{(k, k')}, q_N^{(k)}, q_N^{(k')}\right)\right]\right)^2\right] \leq \frac{C}{\sqrt{N}}$$

and

$$\mathbb{E}\left[\left(\chi\left(q_N^{(k)}\right) - \mathbb{E}\left[\chi\left(q_N^{(k)}\right)\right]\right)^2\right] \leq \frac{C}{\sqrt{N}}.$$

Proof. We decompose

$$\begin{aligned}&\mathbb{E}\left[\left(\psi\left(q_N^{(k, k')}, q_N^{(k)}, q_N^{(k')}\right) - \mathbb{E}\left[\psi\left(q_N^{(k, k')}, q_N^{(k)}, q_N^{(k')}\right)\right]\right)^2\right] \\ &\leq 2\mathbb{E}\left[\left(\psi\left(q_N^{(k, k')}, q_N^{(k)}, q_N^{(k')}\right) - \psi\left(\mathbb{E}\left[q_N^{(k, k')}\right], \mathbb{E}\left[q_N^{(k)}\right], \mathbb{E}\left[q_N^{(k')}\right]\right)\right)^2\right] + \\ &\quad + 2\mathbb{E}\left[\left(\mathbb{E}\left[\psi\left(q_N^{(k, k')}, q_N^{(k)}, q_N^{(k')}\right)\right] - \psi\left(\mathbb{E}\left[q_N^{(k, k')}\right], \mathbb{E}\left[q_N^{(k)}\right], \mathbb{E}\left[q_N^{(k')}\right]\right)\right)^2\right]\end{aligned}$$

using that $(a+b)^2 \leq 2a^2 + 2b^2$. Here the first term on the rhs is $O(N^{-1/2})$ by Lemmata 18 and 21. The second expectation is bounded by the first one by linearity of the expectation and by Jensen's inequality. This implies the first assertion. The second assertion follows analogously. \square

Since all preparations are in place we proceed to the

Proof of Theorem 7. For (26), we use that

$$\begin{aligned} & \mathbb{E} \left[\left(q_N^{(k,k')} - \Psi \left(q_N^{(k-1,k'-1)}, q_N^{(k-1)}, q_N^{(k'-1)} \right) \right)^2 \right] \\ & \leq 9 \mathbb{E} \left[\left(\Psi \left(q_N^{(k,k')}, q_N^{(k)}, q_N^{(k')} \right) - \mathbb{E} \left[\Psi \left(q_N^{(k,k')}, q_N^{(k)}, q_N^{(k')} \right) \right] \right)^2 \right] + 9 \mathbb{E} \left[\left(q_N^{(k,k')} - \mathbb{E} \left[q_N^{(k,k')} \right] \right)^2 \right] \\ & \quad + 9 \left(\mathbb{E} \left[q_N^{(k,k')} \right] - \mathbb{E} \left[\Psi \left(q_N^{(k-1,k'-1)}, q_N^{(k-1)}, q_N^{(k'-1)} \right) \right] \right)^2 \end{aligned}$$

where all terms on the r.h.s. are bounded by a constant times $N^{-1/2}$ by Lemmata 17, 18 and Corollary 22. Assertion (27) follows analogously. \square

6. FIXED POINT ANALYSIS

We recall that q denotes the largest fixed point of ϕ , cfr. (5), defined by:

$$\phi : \mathbb{R}_0^+ \rightarrow [0, 1], \quad x \mapsto \mathbb{E} \tanh^2 (h + \beta \sqrt{xZ}).$$

Theorem 23. For $h \neq 0$ or $h = 0$, $|\beta| \leq 1$

$$\sup_{x \in \mathbb{R}_0^+} \left| \phi^{(k)}(x) - q \right| \rightarrow 0, \text{ as } k \rightarrow \infty.$$

For $h = 0$, $|\beta| > 1$ and $\delta > 0$

$$\sup_{x \geq \delta} \left| \phi^{(k)}(x) - q \right| \rightarrow 0, \text{ as } k \rightarrow \infty.$$

The proof of Theorem 23 proceeds in two main steps. We first show that in a bounded number of steps $\phi^{(k)}$ visits $[q/2, 2q]$. We then establish that $\phi^{(k)}$ contracts strongly on this interval. The case $h = 0$, $|\beta| \leq 1$ needs a slightly different treatment, but turns out to be much simpler.

The way to $[q/2, 2q]$

In this part, we show the following Lemma:

Lemma 24. For $h \neq 0$: $q > 0$ is the unique fixed point of ϕ and we have

$$\sup_{x \in \mathbb{R}_0^+} \inf \{ k \in \mathbb{N} : \phi_\beta^{(k)}(x) \in [q/2, 2q] \} < \infty.$$

For $h = 0$, $|\beta| > 1$: The fixed points of ϕ are 0 and $q > 0$; and for any $\delta > 0$

$$\sup_{x \geq \delta} \inf \{ k \in \mathbb{N} : \phi_\beta^{(k)}(x) \in [q/2, 2q] \} < \infty.$$

For $h = 0$, $|\beta| \leq 1$: $q = 0$ is the unique fixed point of ϕ and we have

$$\sup_{x \in \mathbb{R}_0^+} \left| \phi^{(k)}(x) - q \right| \rightarrow 0, \text{ as } k \rightarrow \infty.$$

The first step to establish Lemma 24 is to prove the following monotonicity property.

Lemma 25. For $h \neq 0$ the fixed point q is unique and strictly positive; and we have

$$\phi(x) > x \text{ if } x \in [0, q) \quad \text{and} \quad \phi(x) < x \text{ if } x > q$$

For $h = 0, |\beta| > 1$ the fixed points of ϕ are 0 and $q > 0$; for $h = 0, |\beta| \leq 1$ the only fixed point is $q = 0$ and in either case we have

$$\phi(x) > x \text{ if } x \in (0, q) \quad \text{and} \quad \phi(x) < x \text{ if } x > q$$

Proof. We refer to the function ϕ for $\beta = 1$ by ϕ_1 . By Proposition A.14.1. of [20] for $\phi = \tanh$ we have that $\phi_1(x)/x$ is a strictly decreasing function on \mathbb{R}^+ . Setting $x = \beta^2 y$ for $\beta \neq 0$ we obtain the same monotonicity for $\phi(y)/y$. For $h \neq 0$ we have $\phi(0) > 0$, hence $\phi(x)/x = 1$ has a unique solution $q > 0$ and $\phi(x)/x < \phi(q)/q = 1$ if and only if $x > q$. For $h = 0$ we have $\phi(0) = 0$. If $\lim_{x \rightarrow 0^+} \phi(x)/x \leq 1$, then $\phi(x)/x < 1$ for $x > 0$ and $q = 0$ is the only fixed point. Otherwise $\lim_{x \rightarrow 0^+} \phi(x)/x > 1$, then $q > 0$ and the claim follows for $x > 0$ the same way it did in the $h \neq 0$ case. Finally we verify the transition for $h = 0$ at $\beta = 1$: By L'Hospital rule $\lim_{x \rightarrow 0^+} \phi(x)/x = \lim_{x \rightarrow 0^+} \phi'(x) = \beta^2$, since

$$\phi'(x) = \beta \mathbb{E}_Z \left[\frac{Z}{2\sqrt{x}} (\tanh^2)'(\beta \sqrt{x}Z) \right] = \frac{\beta^2}{2} \mathbb{E}_Z [(\tanh^2)''(\beta \sqrt{x}Z)]$$

and $(\tanh^2)''(0) = 2(1 - 4\tanh^2(0) + 3\tanh^4(0)) = 2$. \square

We proceed to work out the consequences of such monotonicity.

Lemma 26. Let $I \subset \mathbb{R}$ be a compact interval and $f : I \rightarrow \mathbb{R}$ a continuous function with either $f(x) > x$ for all $x \in I$ or $f(x) < x$ for all $x \in I$, then

$$\sup_{x \in I} \inf \{k \in \mathbb{N} : f^{(k)}(x) \notin I\} < \infty.$$

Proof. By continuity of f , and compactness of I we have that $x \mapsto f(x) - x$ attains its minimum on I . Using that $f(x) > x$ said minimum is strictly positive. Therefore it takes at most $\text{diam}(I)(\min_{y \in I} (f(y) - y))^{-1} < \infty$ many steps for $f^{(k)}(x)$ to leave I . The case $f(x) < x$ follows by considering $-I$ and $-x$. \square

We finish preparations for the proof of Lemma 24 by establishing the following peculiar property of ϕ .

Lemma 27.

$$\phi([q, \infty)) \subset [q/2, 1]$$

Proof. Clearly since $|\tanh| \leq 1$ we have $\phi(x) \leq 1$ by construction. Due to symmetries we can assume $h, \beta \geq 0$ without loss of generality. Using symmetry and monotonicity of \tanh^2 and the Gaussian density around 0

$$\mathbb{E}[\tanh^2(h + \beta\sqrt{q}Z)\mathbf{1}_{Z>0}] \geq \mathbb{E}[\tanh^2(h + \beta\sqrt{q}Z)\mathbf{1}_{Z<0}].$$

Hence for $x \geq q$

$$\frac{q}{2} = \frac{1}{2}\phi(q) \leq \mathbb{E}[\tanh^2(h + \beta\sqrt{q}Z)\mathbf{1}_{Z>0}] \leq \mathbb{E}[\tanh^2(h + \beta\sqrt{x}Z)\mathbf{1}_{Z>0}] \leq \phi(x).$$

\square

Proof of Lemma 24. We first tackle the cases where $q > 0$. Without loss of generality we assume $x \in [0, 1]$, since otherwise we may consider $\phi^{(k)}(x)$ as $\phi^{(k-1)}(\phi(x))$, i.e. shifting the iteration index by one and starting in $\phi(x) \leq 1$ instead of x . By Lemmata 25 and 26 the number of steps needed for $\phi^{(k)}$ to leave $[0, q/2]$ for $h \neq 0$ or $[\delta, q/2]$, for $h = 0, |\beta| >$

1, $0 < \delta < q/2$ is uniformly bounded. After these steps we are either on $[q/2, 2q]$, in which case there is nothing to prove, or we are on $(2q, 1]$. By Lemmata 25 and 26 the iteration leaves $[2q, 1]$ in a uniformly bounded number of steps. Finally applying Lemma 27 yields the claim, since the last point in $[2q, 1]$ before leaving the interval is mapped to $[q/2, 2q]$. Finally for $h = 0, \beta \leq 1$ we have $q = 0$, hence by Lemma 25 the sequence $\phi^{(k)}(x)$ is monotonically decreasing regardless of the starting value. Applying Lemma 26 we obtain that for any $\delta > 0$ the sequence $\phi^{(k)}(x)$ exits $[\delta, 1]$ in a bounded number of steps, uniformly over all starting points $x \in [0, 1]$. The claim follows. \square

Strong contractivity on $[q/2, 2q]$

To finish the proof of Theorem 23 we use the reduction to $[q/2, 2q]$ established in the previous Section and then argue on this interval similarly to the Banach fixed point theorem. To this end we first establish a bound on the derivative of ϕ in Lemma 28, which together with Lemma 25 will give strong estimates for ϕ in Lemma 29:

Lemma 28. For $x > 0$

$$\phi'(x) > -\frac{\phi(x)}{2x}$$

Lemma 29. For any h, β we have

$$q - \frac{1}{2}(x - q) < \phi(x) < x \quad \forall x \in (q, 2q] \quad (73)$$

$$x < \phi(x) < q + \frac{1}{3\frac{x}{q} - 1}(q - x) \quad \forall x \in [q/2, q) \quad (74)$$

Proof of Lemma 28. We first tackle the case $\beta = 1$ and generalize thereafter. Computing the derivative we have for $x \geq 0$

$$\begin{aligned} x\phi_1'(x) &= \int \tanh^2(z)x \left(-\frac{1}{2x} + \frac{(z-h)^2}{2x^2} \right) \frac{1}{\sqrt{2\pi x}} \exp\left(-\frac{(z-h)^2}{2x}\right) dz \\ &= \mathbb{E} \left[\tanh^2(h + \sqrt{x}Z) \frac{Z^2 - 1}{2} \right] > -\frac{1}{2}\phi_1(x). \end{aligned}$$

The last step using that $\mathbb{E}[Z^2 \tanh^2(h + \sqrt{x}Z)] > 0$. The claim now follows for general β , since

$$\phi'(x) = \beta^2 \phi_1'(\beta^2 x) > -\frac{\phi_1(\beta^2 x)}{2x} = -\frac{\phi(x)}{2x}.$$

\square

Proof of Lemma 29. First note that there is no claim if $q = 0$, hence we can assume $q > 0$. The estimates comparing $\phi(x)$ to x follow immediately from Lemma 25. We now prove the remaining two estimates. Clearly for $x \in [q/2, 2q]$

$$\phi(x) - q = \phi(x) - \phi(q) = \int_q^x \phi'(z) dz. \quad (75)$$

Using Lemma 28 and that $\phi(z)/z$ is decreasing (see proof of Lemma 25) we have for $x > q$ and $z \in [q, x]$

$$\phi'(z) > -\frac{\phi(z)}{2z} > -\frac{\phi(q)}{2q} = -\frac{1}{2}.$$

estimating (75) with the above yields (73). By the same argument we obtain for $x \in [q/2, q]$ and $z \in [x, q]$ that

$$\phi'(z) > -\frac{\phi(z)}{2z} > -\frac{\phi(x)}{2x}.$$

Estimating (75) this way yields

$$\phi(x) < q + \frac{q-x}{2x}\phi(x),$$

which can be rearranged to obtain (74)

$$\phi(x) < q\left(1 - \frac{q-x}{2x}\right)^{-1} = q + \frac{1}{3\frac{x}{q} - 1}(q-x).$$

□

Proof of Theorem 23. By Lemma 24 we may assume $q > 0$ (as the claim for $q = 0$ is covered by Lemma 24) and that we start in $[q/2, 2q]$, since otherwise there exists a uniformly bounded number of steps, such that we arrive in this interval. We now iterate one more step and consider

$$a := \min \phi([q/2, 2q]), \quad b := \max \phi([q/2, 2q]).$$

Note that ϕ is a continuous function and therefore attains maximum and minimum on a compact. By Lemma 29 we verify $q/2 < a \leq b < 2q$, which is seen by estimating all x by worst case in the given range. With this we established that $\phi^{(k)}(x)$ enters $[a, b]$ in a uniformly bounded number of steps and never leaves thereafter. We now consider a slightly larger interval of the shape $[q - \gamma, q + 2\gamma]$ to start and choose $\gamma = \max\{q - a, \frac{b-q}{2}\} < q/2$, such that $[a, b] \subset [q - \gamma, q + 2\gamma] \subset (q/2, 2q)$. For $x \in [q - \gamma, q]$ we have $3\frac{x}{q} - 1 \geq 3\frac{a}{q} - 1 > \frac{1}{2}$, where we note that $q - \gamma > q/2$. Hence by Lemma 29 we have

$$x < \phi(x) < q + 2(1 - \varepsilon)(q - x) \quad \forall x \in [q - \gamma, q], \quad (76)$$

where $\varepsilon := 1 - (6\frac{q-\gamma}{q} - 2)^{-1} > 0$. For $d \in (0, \gamma]$, we have $q - \gamma \leq q - d < q + 2d \leq q + 2\gamma$. Hence, by (76) and the first inequality of Lemma 29 we have

$$\phi([q - d, q]) \subset (q - d, q + 2(1 - \varepsilon)d), \quad (77)$$

$$\phi([q, q + 2d]) \subset (q - d, q + 2d). \quad (78)$$

We now show that a uniformly bounded amount of steps suffices for $\phi^{(k)}(x)$ to reach any fixed neighborhood of q . This implies the claim by considering a sequence of smaller and smaller neighborhoods of form $[q - d, q + 2d]$, since the iteration stays in such intervals indefinitely due to (77) and (78). We now fix an arbitrary $d \in (0, \gamma]$. The iteration leaves $[q - d, q - (1 - \varepsilon)d]$ in a uniformly bounded number of steps by Lemmata 26 and 25. Hence the iteration exits $[q - d, q - (1 - \varepsilon)d]$ to a point on $[q - (1 - \varepsilon)d, q + 2(1 - \varepsilon)d]$ and then never escapes this interval: to see this, we use (77) and (78) with $(1 - \varepsilon)d$ in place of d . Finally the iteration also leaves $[q + 2(1 - \varepsilon)d, q + 2d]$ in uniformly boundedly many steps to either $[q - (1 - \varepsilon)d, q + 2(1 - \varepsilon)d]$, in which it stays, or to $[q - d, q - (1 - \varepsilon)d]$, from which it enters $[q - (1 - \varepsilon)d, q + 2(1 - \varepsilon)d]$ in a uniformly bounded amount of steps. Hence for any d there exists a finite $K_d = K_{d,\beta,h} \in \mathbb{N}$, such that

$$\phi^{(K_d)}([q - d, q + 2d]) \subset [q - (1 - \varepsilon)d, q + 2(1 - \varepsilon)d].$$

Repeating this process we see that for any fixed $l \in \mathbb{N}$ the iteration reaches $[q - (1 - \varepsilon)^l \gamma, q + 2(1 - \varepsilon)^l \gamma]$ in a number of steps bounded uniformly over all choices of starting point $x \in \mathbb{R}_0^+$ for $h \neq 0$ and $x > \delta$ for $h = 0$. The claim follows. \square

7. PROOF OF THEOREM 3

Proof of Theorem 3: The case $k = k'$. Let $\varepsilon > 0$, we have

$$\mathbb{P} \left(\left| q_N^{(k)} - q \right| > \varepsilon \right) \leq \mathbb{P} \left(\left| q_N^{(k)} - \phi^{(k-1)} \left(q_N^{(1)} \right) \right| > \frac{\varepsilon}{2} \right) + \mathbb{P} \left(\left| \phi^{(k-1)} \left(q_N^{(1)} \right) - q \right| > \frac{\varepsilon}{2} \right). \quad (79)$$

To show assertion (16) for $k = k'$, it suffices to prove that the $\lim_{k \rightarrow \infty} \limsup_{N \rightarrow \infty}$ of the terms on the r.h.s. of (79) tends to 0. By the Markov inequality, the first term on the r.h.s. is up to the factor $4/\varepsilon^2$ at most

$$\begin{aligned} \mathbb{E} \left[\left(q_N^{(k)} - \phi^{(k-1)} \left(q_N^{(1)} \right) \right)^2 \right] &= \mathbb{E} \left[\left(\sum_{i=0}^{k-2} \phi^{(i)} \left(q_N^{(k-i)} \right) - \phi^{(i+1)} \left(q_N^{(k-i-1)} \right) \right)^2 \right] \\ &\leq k \sum_{i=0}^{k-2} \beta^{4i} \mathbb{E} \left[\left(\phi^{(i)} \left(q_N^{(k-i)} \right) - \phi^{(i+1)} \left(q_N^{(k-i-1)} \right) \right)^2 \right] \quad (80) \\ &\leq k \sum_{i=0}^{k-2} \mathbb{E} \left[\left(q_N^{(k-i)} - \phi^{(1)} \left(q_N^{(k-i-1)} \right) \right)^2 \right] = O \left(\frac{1}{\sqrt{N}} \right) \end{aligned}$$

the second line by $(\sum_{i=1}^k a_i)^2 \leq k \sum_{i=1}^k a_i^2$, the third line using the fact that $\phi(t)$ is β^2 -Lipschitz multiple times. For the last estimate in (80), we used Theorem 7 recognizing

$$\phi \left(q_N^{(k-i-1)} \right) \equiv \psi \left(q_N^{(k-i-1)}, q_N^{(k-i-1)}, q_N^{(k-i-1)} \right).$$

If $h \neq 0$ or $h = 0$ and $\beta < 1$, we bound

$$\lim_{k \rightarrow \infty} \limsup_{N \rightarrow \infty} \mathbb{P} \left(\left| \phi^{(k-1)} \left(q_N^{(1)} \right) - q \right| > \frac{\varepsilon}{2} \right) \leq \lim_{k \rightarrow \infty} \mathbb{P} \left(\sup_{x \in \mathbb{R}_0^+} \left| \phi^{(k)}(x) - q \right| > \frac{\varepsilon}{2} \right) = 0$$

for the second term on the r.h.s. of (79), where the last equality follows from Theorem 23.

If $h = 0$ and $\beta \geq 1$ and $\liminf_{N \rightarrow \infty} q_N^{(1)} > \delta$ for some $\delta > 0$ and bound the second term on the r.h.s. of (79) by

$$\lim_{k \rightarrow \infty} \limsup_{N \rightarrow \infty} \mathbb{P} \left(\left| \phi^{(k-1)} \left(q_N^{(1)} \right) - q \right| > \frac{\varepsilon}{2} \right) \leq \lim_{k \rightarrow \infty} \mathbb{P} \left(\sup_{x \geq \delta/2} \left| \phi^{(k-1)}(x) - q \right| > \frac{\varepsilon}{2} \right) = 0, \quad (81)$$

by Theorem 23.

If $h = 0$ and $\beta \geq 1$, with $\lim_{N \rightarrow \infty} q_N^{(1)} = 0$, we replace q with 0 in the beginning of the proof. Then we bound the second term on the r.h.s. of (79) by

$$\mathbb{P} \left(\phi^{(k-1)} \left(q_N^{(1)} \right) > \frac{\varepsilon}{2} \right) = \mathbb{P} \left(\left| \phi^{(k-1)} \left(q_N^{(1)} \right) - \phi^{(k-1)}(0) \right| > \frac{\varepsilon}{2} \right) \leq \mathbb{P} \left(\beta^{2(k-1)} \left| q_N^{(1)} - 0 \right| > \frac{\varepsilon}{2} \right) \quad (82)$$

using the that ϕ is β^2 -Lipschitz and that 0 is a fixed point. The r.h.s. is tending to 0 as $N \rightarrow \infty$ by assumption. \square

To consider the limiting covariances we introduce the convenient notation

$$\varphi : [-q, q] \mapsto [-q, q], t \rightarrow \psi(t, q, q).$$

Lemma 30. *Let $h = 0$. Then the AT condition (4) is equivalent to $|\beta| \leq 1$.*

Proof. Clearly, (4) is satisfied if $\beta \leq 1$ as $h = 0$. We now assume that $h = 0$ and $\beta > 1$. Then $q > 0$, $\varphi(0) = \mathbb{E} [\tanh(\beta\sqrt{q}Z')]^2 = 0$, and $\varphi(q) = \phi(q) = q$. By the proof of Lemma 2.2 a) of [5], $\varphi'(t) > 0, \varphi''(t) \geq 0$ for $t \geq 0$. As $q > 0$ is a fixed point of φ , it follows that there exists $x \in (0, q)$ with $\varphi(x) < x$. Hence, $\varphi(q) - \varphi(x) > q - x$ and

$$\frac{1}{q-x} \int_x^q \varphi'(t) dt > 1.$$

This in turn implies that $\varphi'(t) > 1$ for some $t \in (x, q)$. It follows that $\varphi'(q) > 1$ as φ' is increasing by Lemma 2.2 a) of [5]. By Lemma 2.2 b) of [5], $\varphi'(q) = \beta^2 \mathbb{E} \cosh^{-4}(h + \beta\sqrt{q}Z)$, hence the AT condition (4) cannot be satisfied. \square

Before we show the remaining part of Theorem 3 some final preparation is needed. We show negative covariances increase when propagated, i.e.

Lemma 31. *For $\beta, h \in \mathbb{R}$ and $t < 0$ we have*

$$\varphi(t) \geq \chi(q)^2 + t,$$

where $\chi(q)^2 = 0$ if and only if $h=0$.

Thus φ has no fixed points on $[-1, 0]$ for $h \neq 0$.

Proof. Fix $\beta \in \mathbb{R}, h \neq 0 \in \mathbb{R}$. There exists only one $q = q_{\beta, h}$ that solves (5). In this proof, q will be considered as a constant and not a function of h anymore. We recall that $\varphi(t) = \psi(\max(-q, \min(t, q)), q, q)$. We will compare this function with

$$\Psi(t) = \mathbb{E} [\tanh(\beta Z) \tanh(\beta Z')] \quad (83)$$

where the expectation is over centered Gaussians Z', Z'' such that

$$\text{Var}(Z) = q, \quad \text{Var}(Z') = q, \quad \text{Cov}(Z, Z') = \min\{\max\{t, -q\}, q\}. \quad (84)$$

Note that for $-q \leq t \leq q$, we can write Ψ as

$$\Psi(t) = \mathbb{E} \left[\tanh \left(\beta \left(\text{sign}(t) \sqrt{|t|} Z + \sqrt{q-|t|} Z' \right) \right) \tanh \left(\beta \left(\sqrt{|t|} Z + \sqrt{q-|t|} Z'' \right) \right) \right] \quad (85)$$

where the expectation is over independent centered Gaussians Z, Z', Z'' . Note that Ψ is an odd function. We claim for the moment that for $t \in [-q, 0]$ and $h \neq 0$, it holds

$$\varphi'(t) \leq \Psi'(t). \quad (86)$$

It follows that

$$\begin{aligned} \varphi(t) &= \varphi(0) - \int_t^0 \varphi'(x) dx \geq \varphi(0) - \int_t^0 \Psi'(x) dx \\ &= \varphi(0) - \Psi(0) + \Psi(t) = \chi(q)^2 + \Psi(t), \end{aligned}$$

since $\varphi(0) = \chi(q)^2 > 0$ and $\Psi(0) = 0$. We will now prove that $\Psi(t) \geq t$ for $t \in [-q, 0]$. For $t \in [0, q]$, we easily check that $\Psi' \geq 0$ and $\Psi'' \geq 0$ (see [5], Lemma 2.2). Thus Ψ is convex, increasing on $[0, q]$ and we claim that

$$\Psi(q) = \mathbb{E} [\tanh^2(\beta\sqrt{q}Z)] \leq \mathbb{E} [\tanh^2(h + \beta\sqrt{q}Z)] = q, \quad (87)$$

(We will see that the proof of this claim is almost the same as the proof of the first claim). In addition, $\Psi(0) = 0$, thus $\Psi(t) \leq t$ on $[0, q]$. As Ψ is odd, we have that $\Psi(t) \geq t$ on $[-q, 0]$. It remains to show (86) and (87). To this end we note that for X, Y independent centered Gaussians with

$$\text{Var}(X) = \beta^2 \frac{q+t}{2} \quad \text{and} \quad \text{Var}(Y) = \beta^2 \frac{q-t}{2}$$

we have the representation

$$\varphi(t) = \mathbb{E}[\tanh(h+X+Y) \tanh(h+X-Y)]$$

and by computing the derivative and applying Gaussian integration by parts

$$\varphi'(t) = \beta^2 \mathbb{E}[\tanh'(h+X+Y) \tanh'(h+X-Y)].$$

We now argue that $\varphi'(t)$ is as a function of h maximized in $h = 0$. This is hardly surprising when noticing that $\tanh' = 1 - \tanh^2$ is maximized in 0, as matching the maximum of \tanh' with the maximum of the Gaussian density produces the largest value. To make this intuition precise we define

$$g(z) := \mathbb{E}[\tanh'(z+Y) \tanh'(z-Y)],$$

so that $\varphi'(t) = \beta^2 \mathbb{E}[g(h+X)]$. Since \tanh' is an even function and Y has the same distribution as $-Y$ one quickly checks that g is an even function as well. Additionally we verify that g is decreasing on \mathbb{R}^+ and increasing otherwise by the following estimate for $0 < z' < z$ or $0 > z' > z$ using the concavity of $\ln \tanh'$:

$$\begin{aligned} g(z) &= \mathbb{E}[\exp(\ln \tanh'(Y-z) + \ln \tanh'(Y+z))] \\ &\leq \mathbb{E}[\exp(\ln \tanh'(Y-z') + \ln \tanh'(Y+z'))] = g(z'). \end{aligned}$$

Writing f_X for the Gaussian density of X we have

$$\varphi'(t) = \int_{-\infty}^{+\infty} g(h+x) f_X(x) dx.$$

g is even, thus g' is an odd function and we have that

$$\begin{aligned} &\frac{d}{dh} \int_{-\infty}^{+\infty} g(h+x) f_X(x) dx \\ &= \int g'(h+x) f_X(x) dx = e^{\frac{-h^2}{2\text{Var}(X)}} \int_{-\infty}^{+\infty} g'(x) f_X(x) e^{\frac{hx}{\text{Var}(X)}} dx \\ &= e^{\frac{-h^2}{2\text{Var}(X)}} \left(- \int_{+\infty}^0 g'(-x) f_X(-x) e^{-\frac{hx}{\text{Var}(X)}} dx + \int_0^{+\infty} g'(x) f_X(x) e^{\frac{hx}{\text{Var}(X)}} dx \right) \\ &= 2e^{\frac{-h^2}{2\text{Var}(X)}} \int_0^{+\infty} g'(x) f_X(x) \sinh\left(\frac{hx}{\text{Var}(X)}\right) dx. \end{aligned} \tag{88}$$

using a linear change of variable twice. We proved that g' is negative on \mathbb{R}^+ and we have that $\text{sign}\left(\sinh\left(\frac{hx}{\text{Var}(X)}\right)\right) = \text{sign}(h)$ for $x > 0$: It follows that

$$h \mapsto \int_{-\infty}^{+\infty} g(h+x) f_X(x) dx$$

is increasing on \mathbb{R}^- , decreasing on \mathbb{R}^+ and thus maximal in $h = 0$. We thus proved (86) and (87) follows by using the same steps as in (88).

We recall that $\chi(q) = \mathbb{E}(\tanh(h + \beta\sqrt{q}Z)) =: f(h)$. Note that $f(0) = 0$ and

$$f'(h) = \mathbb{E}(1 - \tanh^2(h + \beta\sqrt{q}Z)) > 0,$$

which allows us to conclude $\chi(q)^2 = f(h)^2 > 0$ for all $h \neq 0$. \square

We proceed to prove Theorem 3 ($k \neq k'$) in cases. If the AT condition (4) is satisfied $\tilde{q} = q$ and $\varphi(q) = q$ is the unique fixed point of φ . We thereafter give a proof in the case the AT condition (4) is not satisfied. In this case $\tilde{q} < q$ is a second fixed point of φ besides q , which is the fundamental reason a more involved argument is required.

Proof of Theorem 3: The case $k \neq k'$ if the AT condition (4) is satisfied. Under the AT condition (4), Lemma 30 and Theorem 3 for $k = k'$ give $q_N^{(k)} \rightarrow q$ as $N \rightarrow \infty$ followed by $k \rightarrow \infty$ in probability. We claim that

$$\lim_{k \rightarrow \infty} \sup_{t \in [-q, q]} |\varphi^{(k)}(t) - q| = 0 \quad (89)$$

under the AT condition (4). We first finish the proof assuming claim (89) and thereafter return to prove it. Note that

$$|q_N^{(k, k')}| \leq \sqrt{q_N^{(k)} q_N^{(k')}}$$

by definition of these quantities and by the Cauchy-Schwarz inequality. It follows for each $\delta > 0$ that

$$\lim_{k, k' \rightarrow \infty} \limsup_{N \rightarrow \infty} \mathbb{P}(|q_N^{(k, k')}| \geq q + \delta) = 0.$$

Hence,

$$q_N^{(k, k')} - \psi(q_N^{(k-1, k'-1)} \wedge q, q_N^{(k-1)}, q_N^{(k'-1)}) \rightarrow 0$$

in probability as $N \rightarrow \infty$ followed by $k, k' \rightarrow \infty$ by Theorem 7. As ψ is Lipschitz (by Lemma 21) and as $q_N^{(k)} \rightarrow q$ in probability,

$$\psi(q_N^{(k-1, k'-1)} \wedge q, q_N^{(k-1)}, q_N^{(k'-1)}) - \psi(q_N^{(k-1, k'-1)} \wedge q, q, q) \rightarrow 0$$

as $N \rightarrow \infty$ followed by $k, k' \rightarrow \infty$. The second term in the last display equals $\varphi(q_N^{(k-1, k'-1)} \wedge q)$. Hence,

$$q_N^{(k, k')} - \varphi(q_N^{(k-1, k'-1)} \wedge q) \rightarrow 0$$

as $N \rightarrow \infty$ followed by $k, k' \rightarrow \infty$. Iterating these steps and using that φ is Lipschitz, we obtain

$$q_N^{(k, k')} - \varphi^{(\ell)}(q_N^{(k-\ell, k'-\ell)} \wedge q) \rightarrow 0 \quad (90)$$

as $N \rightarrow \infty$ followed by $k, k' \rightarrow \infty$, for any fixed $\ell \in \mathbb{N}$. The assertion follows from (90) and (89), by considering large ℓ .

It remains to show the claim (89). This follows as in Lemma 2.4 b) of [5]. For $h = 0$, the AT condition (4) and Lemma 30 imply that $\beta \leq 1$, and hence that $q = 0$ is the unique fixed point of (5) and $\varphi \equiv 0$. We assume $h \neq 0$ from here on, which implies $q > 0$. Let $\varepsilon \in (0, q)$. From (5), we note that $\varphi(q) = q$. By the proof of Lemma 2.2 a) of [5], $\varphi' > 0$ and $\varphi'' > 0$ on $[0, q]$. Under the AT condition (4), we have $0 \leq \varphi'(q - \varepsilon) < \varphi'(q) \leq 1$ by Lemma 2.2b of [5]. Consequently for $t \in [0, q - \varepsilon]$ there exists $\delta = \delta_\varepsilon > 0$, such that $\varphi'(t) \leq 1 - \delta$ and therefore

$$q \geq \varphi(t) \geq \varphi(q - \varepsilon) - (q - \varepsilon - t)\varphi'(q - \varepsilon) \geq (q - \varepsilon) - (q - \varepsilon - t)(1 - \delta) = t + \delta(q - \varepsilon - t).$$

Hence we obtain uniform convergence of $\varphi^{(k)}$ to q on $[0, q]$, since for any fixed $\varepsilon > 0$ a finite number ($\lceil \delta^{-1} \varepsilon^{-1} \rceil$) of steps suffices to ensure that $\varphi^{(k)}$ is within 2ε range of q . For $t \in [-q, 0]$, Lemma 31 ensures that $\varphi^{(k)}(t) > 0$ after a finite number of steps k and we can conclude the proof if the AT condition (4) is satisfied. \square

*Proof of Theorem 3: The case $k \neq k'$ if the AT condition (4) is **not** satisfied.* For $h = 0$, we have $\tilde{q} = 0$ and all $q_N^{(k,1)}$, $k > 1$ converge to zero by Theorem 7 (27) due to $\chi \equiv 0$. This implies convergence to zero of all $q^{(k,k')}$ for $k \neq k'$ by Theorem 7 (26), since $\psi(0, x, y) = 0$ for $h = 0$. Next for arbitrary $h \in \mathbb{R}$ and $\beta = 0$ we have $\psi \equiv \tanh(h)^2$, $\tilde{q} = q = \tanh^2(h)$ and $m_i^{(k)} = \tanh(h)$ for any $k > 1$, which yields the claim. From here on let $h \neq 0$ and $\beta \neq 0$. We define for $q^{(1)} \in [0, 1]$, $\bar{m} \in [-\sqrt{q^{(1)}}, \sqrt{q^{(1)}}]$, for $k \geq k' \geq 1$

$$q^{(k+1, k'+1)} := \psi(q^{(k, k')}, q^{(k, k)}, q^{(k', k')}), \quad q^{(k+1, 1)} := \chi(q^{(k, k)})\bar{m}.$$

We again shorten $q^{(k, k)}$ to $q^{(k)}$. For the choice $q^{(1)} = q_N^{(1)}$ and $\bar{m} = N^{-1} \sum_i m_i^{(1)}$ by Theorem 7, $q^{(k, k')} - q_N^{(k, k')}$ converges to zero in probability in the large N limit. Note that $q^{(k, k')}$ depends on N only through the N dependent choice of starting condition. It is necessary to make sure that the iteration never fixes to the unstable fixed point $\psi(q, q, q) = q$. To this end we prove

Lemma 32. *For $\beta, h \neq 0$ uniformly over all $q^{(1)}, \bar{m}$ for each $K > 1$ there exists $\delta_K > 0$, such that*

$$\frac{q^{(k, k')}}{\sqrt{q^{(k)} q^{(k')}}} < 1 - \delta_K \text{ for all } k > k', k' \in \{2, \dots, K\}, q^{(1)} \in [0, 1] \text{ and } \bar{m} \in [-\sqrt{q^{(1)}}, \sqrt{q^{(1)}}]. \quad (91)$$

Proof. To obtain the required uniformity over k we consider the following iteration started in $a^{(1)} \in [0, 1], b^{(0)} \in [\min \phi, 1], |\bar{m}| \leq \sqrt{a^{(1)}}$: For $\ell \geq 1$ let

$$a^{(\ell+1)} := \phi(a^{(\ell)}), \quad b^{(\ell)} := \phi(b^{(\ell-1)}) \\ c^{(\ell+1)} := \psi(c^{(\ell)}, a^{(\ell)}, b^{(\ell)}), \quad c^{(1)} := \chi(b^{(0)})\bar{m}.$$

With this we have for $k+1 > k' > 1$, that $q^{(k, k')} = c^{(k')}$ when choosing $a^{(1)} = q^{(1)}$ and $b^{(0)} = q^{(k-k')}$. Note that $k+1 > k'$ is required to ensure that $k - k' \geq 2$, which is needed to have $b^{(0)} \geq \min \phi$. The case $k+1 = k'$ will be resolved later. We now view a, b, c as sequences of continuous functions of the starting values $a^{(1)}, b^{(0)}, \bar{m}$. Hence it suffices to show for $l > 1$ and any starting values that

$$c^{(\ell)} < \sqrt{a^{(\ell)} b^{(\ell)}}, \quad (92)$$

as compactness of the space of starting values and continuity then implies the claim. Note that $c^{(\ell)} \leq \sqrt{a^{(\ell)} b^{(\ell)}}$ is an immediate consequence of Cauchy-Schwarz inequality:

$$\psi(t, x, y) \leq \sqrt{\psi(x, x, x) \psi(y, y, y)} = \sqrt{\phi(x) \phi(y)}.$$

For equality to hold $t = x = y$ is required on the set $\{(t, x, y) : x, y \in [0, 1], t \in [-\sqrt{xy}, \sqrt{xy}]\}$, since $\phi(x) > 0$ due to $h \neq 0$. Hence $c^{(\ell+1)} = \sqrt{a^{(\ell+1)} b^{(\ell+1)}}$ can only happen if $c^{(\ell)} = a^{(\ell)} = b^{(\ell)}$. Propagating this downwards we see that $c^{(\ell)} = \sqrt{a^{(\ell)} b^{(\ell)}}$ for some $\ell \geq 1$ implies that $c^{(1)} = a^{(1)} = b^{(1)}$. Now

$$|c^{(1)}| = |\chi(b^{(0)})\bar{m}| \leq \sqrt{\phi(b^{(0)}) a^{(1)}} = \sqrt{b^{(1)} a^{(1)}},$$

where equality is only possible if $b^{(0)} = 0$, as Jensen is only sharp for almost sure constants. This is a contradiction to $b^{(0)} \geq \min \phi > 0$ due to $h \neq 0$ and therefore concludes the prove for $k+1 > k \geq 1$. We now consider $q^{(k+1,k)}$ as a function of the starting values $q^{(1)}$ and \bar{m} . Following the same line of reasoning we have for $k > 1$ that $q^{(k+1,k)} = \sqrt{q^{(k+1)}q^{(k)}}$ requires $q^{(k,k-1)} = q^{(k)} = q^{(k-1)}$. This however implies that $q^{(k)} = \phi(q^{(k-1)}) = \phi(q^{(k)})$ and therefore $q^{(k)} = q^{(k-1)} = q$. Propagating this down again we obtain $q^{(1)} = q$, which contradicts that equality in $|q^{(2,1)}| = |\chi(q^{(1)})\bar{m}| \leq \sqrt{\phi(q^{(1)})q^{(1)}}$ only holds for $q^{(1)} = 0$, again due to Jensen being only sharp for almost sure constants. Hence $q^{(k+1,k)} < \sqrt{q^{(k+1)}q^{(k)}}$. Again for $k > 1$ the claim follows due $\frac{q^{(k+1,k)}}{\sqrt{q^{(k+1)}q^{(k)}}}$ being a continuous function of the starting values, that is defined on a compact, hence a maximum is achieved, which strictly below one. \square

From here on, we only consider the choice $q^{(1)} = q_N^{(1)}$ and $\bar{m} = N^{-1} \sum_i m_i^{(1)}$ only and no longer consider the iteration as a function of starting values. Next we define for $k > k' \geq 1$

$$\kappa^{(k+1,k'+1)} := f_{k,k'}(\kappa^{(k,k')}), \text{ and } \kappa^{(k,1)} := \begin{cases} \frac{\chi(q^{(k-1)})\bar{m}}{\sqrt{q^{(k)}q^{(1)}}} & , \text{ if } q^{(1)} > 0 \\ 0 & , \text{ else} \end{cases}, \text{ where}$$

$$f_{k,k'}(x) := \frac{\psi(x\sqrt{q^{(k)}q^{(k')}}), q^{(k)}, q^{(k')}}{\sqrt{\phi(q^{(k)})\phi(q^{(k')})}}.$$

Note that if $q^{(1)} = 0$ the value of $\kappa^{(k,1)}$ plays no role in the propagation, since it is multiplied by zero when computing $\kappa^{(k+1,2)} = \psi(0, 0, q^{(k)}) / \sqrt{\phi(0)\phi(q^{(k)})}$. By Theorem 7 and remembering $\phi(t) \equiv \psi(t, t, t)$ by definition, see (17) and (14), we observe that the correlation functional

$$\kappa_N^{(k,k')} \equiv \frac{q_N^{(k,k')}}{\sqrt{q_N^{(k)}q_N^{(k')}}}$$

satisfies for any $k > k' > 1$

$$|\kappa_N^{(k,k')} - \kappa^{(k,k')}| \rightarrow 0 \text{ as } N \rightarrow \infty$$

in probability. We remark that $\kappa^{(k,k')}$ may depend on N , but only through the starting values $q^{(1)}$ and \bar{m} . Note that $\psi(t, x, y)$ is strictly increasing in $t \in [-\sqrt{xy}, \sqrt{xy}]$ for any $x, y > 0$ (see Lemma 19 with $f = g = \tanh$) and therefore so is $f_{k,k'}$ for any k, k' . Further both functions are Lipschitz by corollary 21. Since $q^{(k)} \rightarrow q$ by Theorem 3 we obtain, using that the Lipschitz constant is uniformly bounded, uniform convergence of $t \mapsto \psi(t, q^{(k)}, q^{(k')})$ to $t \mapsto \psi(t, q, q)$ as well as uniform convergence of $f_{k,k'}$ to $t \mapsto \psi(qt, q, q)/q$. Using that $|\tanh''| \leq 2$ it is simple to check that $\frac{d}{dt}\psi(t, x, y)$ as well as $f'_{k,k'}$ are $4\beta^4$ Lipschitz (see corollary 21). This again gives uniform convergence of $t \mapsto \frac{d}{dt}\psi(t, q^{(k)}, q^{(k')})$ to $t \mapsto \frac{d}{dt}\psi(t, q, q)$ and of $f'_{k,k'}$ to $t \mapsto \frac{d}{dt}(\psi(qt, q, q)/q)$. Corollary 21 gives uniform convergence and bounds for higher derivatives of $f'_{k,k'}$, since all derivatives of \tanh are bounded. By assumption the AT condition (4) is not satisfied, which is equivalent to $\frac{d}{dt}\psi(t, q, q)|_{t=q} > 1$, where the derivative is taken from below or equivalently on the set $A := \{(t, x, y) : t \leq \sqrt{xy}, x, y \in [0, 1]\}$. Since ψ' is continuous on A there exists $\varepsilon > 0$ and an in A open neighborhood U_ε of (q, q, q) such that $\inf \psi'(U_\varepsilon) \geq 1 + \varepsilon$. This implies that

for sufficiently small $\varepsilon > 0$ and sufficiently small $\delta \equiv \delta_\varepsilon > 0$ there exists $K \in \mathbb{N}$, s.t. $f'_{k,k'}(t) > 1 + \varepsilon$ if $k, k' \geq K$ and $t > 1 - \delta$. This implies for $k, k' \geq K$ and $t \in (1 - \delta, 1)$ that

$$f_{k,k'}(t) = f_{k,k'}(1) - \int_t^1 f'_{k,k'}(x)dx \leq 1 - (1-t)(1+\varepsilon) = t - (1-t)\varepsilon. \quad (93)$$

We now recall that \tilde{q} is the smaller of the two fixed points of the on $[0, 1]$ convex and increasing function $t \mapsto \psi(t, q, q)$ and that this function is strictly above the identity on $[-1, \tilde{q})$ and below for (\tilde{q}, q) (see Lemma 31 and Lemma 2.2 a) of [5]). Hence by the established uniform convergence, for any $\delta' > 0$, there exist $\varepsilon' > 0$ and $K' \in \mathbb{N}$, s.t. $f_{k,k'}(t) > t + \varepsilon'$ for $t \in [-1, \tilde{q}/q - \delta']$ and $f_{k,k'}(t) < t - \varepsilon'$ for $t \in [\tilde{q}/q + \delta', 1 - \delta']$ for any $k, k' \geq K'$. We choose $\delta' = \delta$ for simplicity. With this, we observe for sufficiently large k , using additionally that $f_{k,k'}$ is continuous, increasing and convex on $\frac{\tilde{q}}{q} + [-\delta, \delta]$ due to uniform convergence, that $f_{k,k'}$ has exactly one fixed point on $\tilde{q}/q + [-\delta, \delta]$ as long as δ is chosen smaller than $(1 - \frac{\tilde{q}}{q})/2$. We will refer to this fixed point by $\alpha_{k,k'}$. By (91) $\kappa_{K'+\ell, K'} < 1 - \delta_{K'}$ for any $\ell > 0$. We now see that for large k, k' that $n_{\varepsilon'} = \lceil 1/\varepsilon' \rceil$ steps suffice to ensure that $\kappa_{k+n_{\varepsilon'}, k'+n_{\varepsilon'}} > 0$. Any positive value $t \in [0, \tilde{q}/q - \delta]$ of κ increases by at least ε' each step and eventually enters $\tilde{q}/q + [-\delta, \delta]$ as it can not jump over $\alpha_{k,k'} \in \frac{\tilde{q}}{q} + [-\delta, \delta]$, since $f_{k,k'}(t) \leq f_{k,k'}(\alpha_{k,k'}) = \alpha_{k,k'}$. On the other hand any value $t \in [\tilde{q}/q + \delta, 1 - \delta]$ decreases and lands in $[\tilde{q}/q - \delta, \tilde{q}/q + \delta]$ as well due to the same reasons. Finally values on $[1 - \delta, 1 - \delta_K]$ get repulsed from 1 due to $f_{k,k'}(t) \leq t - (1-t)\varepsilon \leq t - \delta_K\varepsilon$ by (93) and land in $\tilde{q}/q + [-\delta, \delta]$ as well. Hence for $k > k'$ both large enough for any $\delta > 0$ there exists a number of steps n_δ uniformly bounded over all choices of $q^{(1)}, \bar{m}$ such that $\kappa_{k+n_\delta, k'+n_\delta} \in \frac{\tilde{q}}{q} + [-\delta, \delta]$. Since $\tilde{q} > 0$ is the smallest fixed point of the on $[0, q]$ convex function $\psi(t, q, q)$ with $\psi(0, q, q) > 0$ we have $\frac{d}{dt}\psi(t, q, q)|_{t=\tilde{q}} < 1$. Since $f_{k,k'}$ is increasing and above (below) the identity before (after) $\alpha_{k,k'}$. This implies that $f_{k,k'}(t)$ is always in between t and $\alpha_{k,k'}$. Since $\alpha_{k,k'} \in \tilde{q}/q + [-\delta, \delta]$ we have established that the iteration never leaves $\tilde{q}/q + [-\delta, \delta]$ once entered. Considering a sequence of δ approaching zero from above yields convergence of $\kappa_{k,k'}$ to \tilde{q}/q uniformly in the starting values and therefore the claim due to the convergence of $q^{(k)}$ to q . \square

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