

# COMPLEXITY OF LOCAL MAXIMA OF GIVEN RADIAL DERIVATIVE FOR MIXED $p$ -SPIN HAMILTONIANS

DAVID BELIUS, MARIUS A. SCHMIDT

*Department of Mathematics and Computer Science, University of Basel, Switzerland*

ABSTRACT. We study the number of local maxima with given radial derivative of spherical mixed  $p$ -spin models and prove that the second moment matches the square of the first moment on exponential scale for arbitrary mixtures and any radial derivative. This is surprising, since for the number of local maxima with given radial derivative and given energy the corresponding result is only true for specific mixtures [Sub17; BSZ20].

We use standard Kac-Rice computations to derive formulas for the first and second moment at exponential scale, and then find a remarkable analytic argument that shows that the second moment formula is bounded by twice the first moment formula in this general setting.

This also leads to a new proof of a central inequality used to prove concentration of the number critical points of pure  $p$ -spin models of given energy in [Sub17] and removes the need for the computer assisted argument used in that paper for  $3 \leq p \leq 10$ .

## 1. INTRODUCTION

The spherical mixed  $p$ -spin models are a natural and general class of isotropic differentiable Gaussian random fields on the sphere. They are paradigmatic models of high-dimensional complex random landscapes and originate in spin glass theory [SK75; Der80; GM84; Tal00; KTJ76; CS92; Tal06; CL04; MPV87; Tal10; Pan13]. We study the number of local maxima of the field with fixed radial derivative. Specifically, we compute the second moment, and show that on an exponential scale it matches the first moment *for any mixed  $p$ -spin model and any radial derivative*. This strongly suggests, but does not yet prove, that the number of local maxima of given radial derivative concentrates around its mean. Expressed in the spin glass terminology it thus strongly suggests that quenched and annealed complexity of local maxima of given radial derivative *always* coincides.

This is surprising, since for the previously studied number of local maxima (or critical points) at fixed radial derivative and fixed energy this is only true for all radial derivatives and energies for very special mixed  $p$ -spin models, namely the pure  $p$ -spin models and their perturbations [ABC13; AB13; Sub17; BSZ20].

To state our result, consider a mixed  $p$ -spin Hamiltonian  $H_N$  with mixture  $\xi$  on the unit sphere  $S_{N-1} \subset \mathbb{R}^N$ , i.e  $H_N$  is a centered Gaussian field on  $S_{N-1}$  with covariance given by

$$\text{Cov}[H_N(\sigma), H_N(\tau)] = N\xi(\sigma\tau),$$

for  $\sigma, \tau \in S_{N-1}$  and  $\sigma\tau = \sum_{i=1}^N \sigma_i \tau_i$  the inner product. Any isotropic centered Gaussian field on the sphere must have a covariance of this form, and a  $\xi$  gives a well-defined covariance for all  $N$  if and only if it is of the form  $\xi(x) = \sum_{p \geq 0} a_p x^p$  for  $p \geq 0$  with  $\xi(1) < \infty$  [Sch42]. Our

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*E-mail address:* david.belius@cantab.net, m.schmidt@mathematik.uni-frankfurt.de.

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results apply to any  $\xi$  of such a form with  $a_0 = a_1 = 1$  and radius of convergence greater than 1, which is thus a very general class of isotropic Gaussian fields on the sphere. For these  $\xi$  the field  $H_N$  is almost surely smooth on a ball of radius larger than one (see Lemmata A.1 and A.7 [Bel22]). For  $\partial_r$  the derivative in radial direction the central object of our study is the number

$$\mathcal{N}(D) = \text{Number of local maxima of } H_N \text{ on } S_{N-1} \text{ with } \frac{1}{N} \partial_r H_N \in D.$$

Note that  $H_N$  is defined on a ball so that we may speak of its radial derivative, but we always consider local maxima with respect to the unit sphere. We will use what are by now standard Kac-Rice computations to show that (see Lemma 2.1)

$$(1.1) \quad \frac{1}{N} \ln E[\mathcal{N}(D)] \rightarrow \sup_{x \in D} I(x),$$

where the function  $I$  is given explicitly in terms of the shorthands  $\xi' = \xi'(1)$  and  $\xi'' = \xi''(1)$  by

$$I(x) := \frac{1}{2} \ln \left( \frac{\xi''}{\xi'} \right) - \frac{x^2}{4\xi''} \frac{\xi'' - \xi'}{\xi'' + \xi'} + \Omega \left( \frac{x}{\sqrt{2\xi''}} \right),$$

for

$$(1.2) \quad \Omega(y) := \begin{cases} -\frac{1}{2}y\sqrt{y^2-2} + \ln \left( \frac{y+\sqrt{y^2-2}}{\sqrt{2}} \right) & \text{for } y \geq \sqrt{2}, \\ -\infty & \text{else.} \end{cases}$$

Writing

$$(1.3) \quad r_\infty := \inf\{x \in \mathbb{R} : I(x) \geq 0\} = 2\sqrt{\xi''} \quad \text{and} \quad r_0 := \sup\{x \in \mathbb{R} : I(x) \geq 0\}$$

we note that  $I$  is  $-\infty$  below  $r_\infty$  and strictly decreasing on  $[r_\infty, \infty)$  with  $I(r_0) = 0$ . This entails by Markov inequality that with high probability there are no local maxima of  $H_N$  on  $S_{N-1}$  with radial derivative significantly above  $r_0$  or below  $r_\infty$ . Our main result is the following, which shows the second and first moment match on exponential scale for *any* mixture  $\xi$ .

**Theorem 1.1** (Matching moments on exponential scale). *For all  $x \in [r_\infty, r_0]$  we have*

$$(1.4) \quad \lim_{\varepsilon \searrow 0} \lim_{N \rightarrow \infty} \frac{1}{N} \ln E[\mathcal{N}([x - \varepsilon, x + \varepsilon])^2] - \frac{2}{N} \ln E[\mathcal{N}([x - \varepsilon, x + \varepsilon])] = 0.$$

*Additionally*

$$(1.5) \quad \lim_{\varepsilon \searrow 0} \lim_{N \rightarrow \infty} \mathbb{P}(\mathcal{N}((-\infty, r_\infty - \varepsilon] \cup [r_0 + \varepsilon, \infty)) = 0) = 1.$$

As mentioned above, it is surprising that the matching (1.5) of moments holds *for all*  $\xi$  with  $a_0 = a_1 = 0$  and  $x \in [r_\infty, r_0]$ . Proving the matching of the first and second moment on exponential scale is the main step when proving concentration around the mean using the second moment method. Since this main step is achieved by the our theorem we make the following conjecture.

**Conjecture 1.2.** *For all  $r \in [r_\infty, r_0]$*

$$(1.6) \quad \lim_{\varepsilon \searrow 0} \lim_{N \rightarrow \infty} \frac{1}{N} \ln \mathcal{N}([x - \varepsilon, x + \varepsilon]) = I(x)$$

*in probability.*

In particular, Conjecture 1.2 would imply that the total number of local maxima is in  $\exp(NI(r_\infty) + o(N))$  with high probability for all  $\xi$ .

One way to prove Conjecture 1.2 would be to show that the second moment is asymptotic to the first moment squared, i.e. a stronger version of (1.4). This seems attainable using the methods of [Sub17] for mixtures with  $a_2 = 0$ , but is beyond the scope of this paper<sup>1</sup>. Alternatively, if one could obtain a general concentration result for  $\mathcal{N}$  such as is available e.g. for Lipschitz functions of independent Gaussian random variables then Conjecture 1.2 could be derived from the exponential scale matching of moments (1.4).

To prove Theorem 1.1 we follow [Sub17; BSZ20] by using the Kac-Rice formula to compute a variational formula for the second moment (see Lemma 2.2). The upshot is

$$\frac{1}{N} \ln E[\mathcal{N}(D)^2] \rightarrow \sup \{S(x, \alpha) : \alpha \in (-1, 1), x \in D\},$$

where  $S$  is given by (2.1) below.

It is easily checked that for all  $x \in [r_\infty, r_0]$  one has  $S(x, 0) = 2I(x)$ . The main step of the proof of (1.4) is then an elementary but complicated computation that proves that  $S(x, \alpha)$  is maximized at  $\alpha = 0$ , which with (1.1) gives (1.4). Beyond the fact that this holds for *any* mixture and  $x \in [r_\infty, r_0]$ , it is also pleasantly surprising that an analytic proof of it can be obtained.

For pure  $p$ -spin mixtures, i.e. for  $\xi(x) = x^p$ , a result equivalent to Theorem 1.1 was proved in [Sub17], which studied the number of critical points of pure  $p$ -spin models at fixed energy. The equivalence is due to the fact that at low energies most critical points are local maxima, and that for pure  $p$ -spin models  $\partial_r H_N(\sigma) = p H_N(\sigma)$  for all  $\sigma \in S_{N-1}$  almost surely and therefore a restriction on the radial derivative is equivalent to a restriction on the energy. Note however that for  $3 \leq p \leq 10$  the proof of [Sub17] is computer assisted, as a computer plot is employed to show that the formula corresponding to  $S(x, \alpha)$  is maximized at  $\alpha = 0$  (see [Sub17, Proof of Lemma 7, Figure 1]). Our proof on the other hand is fully analytic for all  $\xi$ .

Since its introduction as a tool for the mathematically rigorous study of spin glasses and mixed  $p$ -spin models [Fyo04; FN12; ABČ13] the use of the Kac-Rice formulas has become standard. Without distinguishing between the different types of critical points that have been considered (all critical points, local maxima, saddles of fixed index, etc.), the first moment for pure mixtures is discussed in [ABČ13; Fyo15], the second moment in [Sub17], the first moment for mixed models in [AB13; Fyo15] and the second in [BSZ20]. As mentioned above, for pure models [Sub17] shows that the complexity concentrates around its mean. The results of [BSZ20] extend this to mixed models that are small perturbation of pure models. The first moment in the presence of external field is computed in [Fyo15; Bel+22].

Before going into details we discuss the structure of the paper. In Section 2 we compute the limits in (1.4) in form of Lemma 2.1 for the first moment and Lemma 2.2 for pairs of given inner product  $\alpha$ , which gives the second moment by optimizing over  $\alpha$ . In Section 3 we compare aforementioned limits for non-negative  $\alpha$  in Theorem 3.1. Finally in Section 4 we combine our insights to prove Theorem 1.1.

## 2. FORMULAS FOR ANNEALED ONE AND TWO POINT COMPLEXITIES

In this section we control the expected exponential rate of the number of local maxima and number of pairs of local maxima (see Lemmata 2.1 and 2.2 below) by adapting results of [BSZ20,

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<sup>1</sup>The restriction  $a_2 = 0$  is to have approximate independence of  $M^{(1)}$  and  $M^{(2)}$  for small  $\alpha$ , see proof of Lemma 2.2, [Sub17, Lemma 13 (4.8) and (2)] and [BSZ20, Lemma 14 (4.10) and (2)].

Theorem 5] from critical points to local maxima. We first state our results and then continue with the needed proofs.

**Lemma 2.1** (First moment). *For any open  $D \subset \mathbb{R}$  with  $2\sqrt{\xi''} \notin \partial D$*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \ln E[\mathcal{N}(D)] = \sup_{x \in D} I(x).$$

To state the result on pairs of local maxima properly consider

$$\mathcal{N}_2(D, A) = \# \left\{ (\sigma_1, \sigma_2) : \sigma_1 \sigma_2 \in A, \sigma_i \text{ loc. max. of } H_N \text{ with } \frac{1}{N} \partial_r H_N(\sigma_i) \in D \text{ for } i \in \{1, 2\} \right\},$$

where ‘‘loc. max.’’ refers to local maxima on the sphere. With  $\xi'_\alpha = \xi'(\alpha)$  and  $\xi''_\alpha = \xi''(\alpha)$  as shorthands we define

$$(2.1) \quad S(x, \alpha) = \frac{1}{2} \ln \left( \frac{(1 - \alpha^2) \xi''^2}{\xi'^2 - \xi_\alpha'^2} \right) + \frac{x^2}{2\xi''} Q(\alpha) + 2 \Omega \left( \frac{x}{\sqrt{2\xi''}} \right),$$

where

$$(2.2) \quad Q(\alpha) = 1 - \frac{2\xi''(\xi' - \alpha\xi'_\alpha + (1 - \alpha^2)\xi''_\alpha)}{\xi'^2 - \xi_\alpha'^2 + (\xi' - \alpha\xi'_\alpha)(\xi'' + \xi''_\alpha) + (1 - \alpha^2)\xi''\xi''_\alpha}$$

which allows us to state our result on pairs of local maxima as follows:

**Lemma 2.2** (Two point annealed complexity at exponential scale). *It holds for any interval  $A \subset (-1, 1)$ , that*

$$\lim_{\varepsilon \searrow 0} \lim_{N \rightarrow \infty} \frac{1}{N} \ln E[\mathcal{N}_2((x - \varepsilon, x + \varepsilon), A)] \leq \sup_{\alpha \in A} S(x, \alpha).$$

The remainder of the section is devoted to the proofs:

**Proof of Lemma 2.1.** We adapt the proof of [BSZ20, Theorem 5] with  $q = 1$ ,  $B = \mathbb{R}$  and consider local maxima instead of critical points. The change to local maxima only causes an extra indicator of the event  $\{\lambda_{\max}(\nabla_{\text{sp}}^2 H_N(\sigma)) < 0\}$  to appear, when using Kac-Rice formula [AT07, Theorem 12.1.1], since a critical point  $\sigma$  on this event is a local maximum and vice versa almost surely. Following the proof up to [BSZ20, (4.2)], conditioning on the radial derivative, but not on the energy, and carrying the indicator along we obtain

(2.3)

$$E[\mathcal{N}(D)] = \exp \left( N \left( \frac{1}{2} + \frac{1}{2} \ln \left( \frac{\xi''}{\xi'} \right) \right) + o(N) \right) \times \\ \times \int_D \exp \left( -N \frac{x^2}{2(\xi' + \xi'')} \right) \mathbb{E} \left[ \left| \det \left( G - \sqrt{\frac{N}{N-1}} \frac{x}{\xi''} I_{N-1} \right) \right| \mathbf{1}_{\{\lambda_{\max}(G) < \sqrt{\frac{N}{N-1}} \frac{x}{\xi''}\}} \right] dx,$$

where  $I_{N-1}$  is the identity matrix of dimension  $N - 1$ ,  $\lambda_{\max}$  is the largest eigenvalue and  $G$  is a normalized GOE matrix of dimension  $N - 1$ , i.e it is real symmetric with otherwise independent centered Gaussian entries of variance  $\frac{1}{N-1}$  off the diagonal and variance  $\frac{2}{N-1}$  on the diagonal. This reduces the problem to computing the  $\frac{1}{N} \ln$  limit of the expectation and applying Laplace principle. Let  $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_{N-1}$  be the eigenvalues of  $G - \sqrt{\frac{N}{N-1}} \frac{x}{\xi''} I_{N-1}$ . If  $x < 2\sqrt{\xi''}$  we have by Cauchy-Schwartz and estimating roughly

$$\mathbb{E} \left[ \prod_{i=1}^{N-1} |\lambda_i| \mathbf{1}_{\{\lambda_{N-1} < 0\}} \right] \leq \mathbb{E}[\lambda_{N-1}^{2(N-1)}]^{1/2} \mathbb{P}(\lambda_{N-1} < 0)^{1/2}.$$

Now by [BDG01, Lemma 6.3] the first term is at most of order  $\exp(\text{const}N)$ , while the second term decays as  $\exp(-\text{const}N^2)$ , since it requires a macroscopic change of the empirical spectral distribution of a GOE matrix which has LDP of rate  $N^2$  see e.g. [BDG01, Theorem 6.1]. Hence  $x < 2\sqrt{\xi''}$  contribute  $-\infty$  to the  $\frac{1}{N} \ln$  limit of the expectation. On the other hand if  $x > 2\sqrt{\xi''}$  we have

$$\mathbb{E} \left[ \prod_{i=1}^{N-1} |\lambda_i| \mathbf{1}_{\{\lambda_{N-1} < 0\}} \right] = \mathbb{E} \left[ \prod_{i=1}^{N-1} |\lambda_i| \right] - \mathbb{E} \left[ \prod_{i=1}^{N-1} |\lambda_i| \mathbf{1}_{\{\lambda_{N-1} \geq 0\}} \right],$$

where by [BSZ20, (4.3)] we have

$$(2.4) \quad \lim_{\epsilon \searrow 0} \lim_{N \rightarrow \infty} \frac{1}{N} \ln \left( \int_{y-\epsilon}^{y+\epsilon} \mathbb{E} \left[ \prod_{i=1}^{N-1} |\lambda_i| \right] dx \right) = \frac{y^2}{4\xi''} - \frac{1}{2} - \frac{y}{4\sqrt{\xi''}} \sqrt{\frac{y^2}{\xi''} - 4} + \ln \left( \sqrt{\frac{y^2}{4\xi''} - 1} + \frac{y}{2\sqrt{\xi''}} \right)$$

and by Cauchy-Schwarz followed by application of [Sub17, Corollary 22 and 23] gives us

$$\mathbb{E} \left[ \prod_{i=1}^{N-1} |\lambda_i| \mathbf{1}_{\{\lambda_{N-1} \geq 0\}} \right] \leq \mathbb{E} \left[ \prod_{i=1}^{N-1} |\lambda_i|^2 \right]^{1/2} \mathbb{P}(\lambda_{N-1} \geq 0)^{1/2} \leq C \mathbb{E} \left[ \prod_{i=1}^{N-1} |\lambda_i| \right] \mathbb{P}(\lambda_{N-1} \geq 0)^{1/2}$$

for some constant  $C > 0$  independent of  $N$ . For  $x > 2\sqrt{\xi''}$  we have by [BDG01, Theorem 6.2] that  $\mathbb{P}(\lambda_{N-1} \geq 0) \rightarrow 0$  and therefore the indicator does not contribute to the limit for  $x > 2\sqrt{\xi''}$ . Collecting cases we have shown that

$$(2.5) \quad \begin{aligned} \lim_{\epsilon \searrow 0} \lim_{N \rightarrow \infty} \frac{1}{N} \ln \left( \int_{y-\epsilon}^{y+\epsilon} \mathbb{E} \left[ \prod_{i=1}^{N-1} |\lambda_i| \mathbf{1}_{\{\lambda_{N-1} < 0\}} \right] dx \right) &= \\ &= \begin{cases} \frac{y^2}{4\xi''} - \frac{1}{2} - \frac{y}{4\sqrt{\xi''}} \sqrt{\frac{y^2}{\xi''} - 4} + \ln \left( \sqrt{\frac{y^2}{4\xi''} - 1} + \frac{y}{2\sqrt{\xi''}} \right) & \text{for } y \geq 2\sqrt{\xi''} \\ -\infty & \text{else} \end{cases} \end{aligned}$$

Applying Laplace principle to (2.3) using (2.5) yields the claim.  $\square$

**Proof of Lemma 2.2.** Since all local maxima are critical points we have for by [BSZ20, Theorem 6] for  $q_1 = q_2 = 1$ ,  $B_1 = B_2 = \mathbb{R}$ ,  $D_1 = D_2 = (x - \epsilon, x + \epsilon)$  and  $I = A$  that

$$\lim_{\epsilon \searrow 0} \limsup_{N \rightarrow \infty} \mathbb{E}[\mathcal{N}_2(D_1, I)] \leq \sup_{\alpha \in A, u_1, u_2 \in \mathbb{R}} \Psi_{\xi, 1, 1}(\alpha, u_1, u_2, x, x),$$

where  $\Psi$  is given in [BSZ20, (3.7)]. The optimization with respect to  $u_1, u_2$ , using Laplace principle on correlated Gaussian tails, yields

$$\sup_{u_1, u_2 \in \mathbb{R}} -\frac{1}{2}(u_1, u_2, x, x) \Sigma_{U, X}^{-1}(\alpha, 1, 1)(u_1, u_2, x, x)^T = -\frac{1}{2}(x, x) \Sigma_X^{-1}(\alpha, 1, 1)(x, x)^T,$$

where by [BSZ20, (A.3)]

$$\Sigma_X(\alpha, 1, 1) = \begin{pmatrix} \xi'' + \xi' - \frac{(\alpha\xi'' + \xi')^2(1-\alpha^2)\xi'}{\xi'^2 - (\alpha\xi'' - \xi')(1-\alpha^2)^2} & \alpha^2\xi'' + \alpha\xi' - \frac{(\alpha\xi'' + \xi')^2(1-\alpha^2)(\alpha\xi' - \xi''(1-\alpha^2))}{\xi'^2 - (\alpha\xi'' - \xi')(1-\alpha^2)^2} \\ \alpha^2\xi'' + \alpha\xi' - \frac{(\alpha\xi'' + \xi')^2(1-\alpha^2)(\alpha\xi' - \xi''(1-\alpha^2))}{\xi'^2 - (\alpha\xi'' - \xi')(1-\alpha^2)^2} & \xi'' + \xi' - \frac{(\alpha\xi'' + \xi')^2(1-\alpha^2)\xi'}{\xi'^2 - (\alpha\xi'' - \xi')(1-\alpha^2)^2} \end{pmatrix}.$$

For  $x \geq 2\sqrt{\xi''}$  we obtain the claim, being careful to not confuse  $\Omega$  in (1.2) with [BSZ20, (3.3)], by verifying that

$$-\frac{1}{2}(x, x) \Sigma_X^{-1}(\alpha, 1, 1)(x, x)^T + \frac{x^2}{2\xi''} = \frac{x^2}{2\xi''} Q(\alpha),$$

which is a straightforward computation using that  $(x, x)\Sigma_X^{-1}(\alpha, 1, 1)(x, x)^T = \frac{2x^2}{\Sigma_X(\alpha, 1, 1)_{1,1} + \Sigma_X(\alpha, 1, 1)_{1,2}}$  and canceling  $\xi' + \alpha\xi'_\alpha - \xi''_\alpha(1 - \alpha^2)$ . It remains to show for  $x < 2\sqrt{\xi''}$  that

$$\lim_{\varepsilon \searrow 0} \limsup_{N \rightarrow \infty} \mathbb{E}[\mathcal{N}_2(D_1, I)] = -\infty.$$

To this end we follow the proof of [BSZ20, Theorem 6] for  $q_1 = q_2 = 1$ ,  $B_1 = B_2 = \mathbb{R}$ ,  $D_1 = D_2 = (x - \varepsilon, x + \varepsilon)$  and insert an additional indicator (with the same argument as in the proof of Lemma 2.1) to adjust for the change from critical points to local maxima. We arrive at [BSZ20, (4.12)], which reads for our case:

$$\mathbb{E}[\mathcal{N}_2(D, A)] = C_N \int_A \mathcal{D}(\alpha)^{N-3} \mathcal{F}(\alpha) \mathbb{E} \left[ \prod_{i=1}^2 \left| \det \left( M_{N-1}^{(i)}(\alpha) \right) \right| \mathbf{1}_{E_i} \right],$$

where

$$C_N = \omega_N \omega_{N-1} \left( \frac{1}{2\pi} (N-1) \frac{\xi''}{\xi'} \right)^{N-1}, \quad \omega_N = 2 \frac{\pi^{N/2}}{\Gamma(N/2)},$$

$$\mathcal{D}(\alpha) = \sqrt{1 - \alpha^2} \sqrt{1 - \frac{\xi'^2}{\xi'^2}},$$

$$\mathcal{F}(\alpha) = \sqrt{1 - \frac{\xi'^2}{\xi'^2}} \sqrt{1 - \left( \frac{\alpha\xi'_\alpha - \xi''_\alpha(1 - \alpha^2)}{\xi'} \right)^2},$$

$$E_i = \{X_i(\alpha) \in \sqrt{N}D, \lambda_{\max}(M_{N-1}^{(i)}(\alpha)) < 0\},$$

and  $X_1(\alpha), X_2(\alpha), M_{N-1}^{(1)}(\alpha), M_{N-1}^{(2)}(\alpha)$  have joint distribution given by [BSZ20, Lemma 15]. The terms  $C_N, \mathcal{D}(\alpha)^{N-3}, \mathcal{F}(\alpha)$  are all bounded by  $\exp(\text{const}N)$  and therefore the claim follows immediately from the Cauchy-Schwartz inequality if we show that

$$\mathbb{E} \left[ \prod_{i=1}^2 \det \left( M_{N-1}^{(i)}(\alpha) \right)^2 \right] \leq \exp(\text{const}N),$$

since by the eigenvalue interlacing theorem  $E_1, E_2$  require a large deviation of the empirical spectral measure of a GOE and therefore by [BDG01, Theorem 6.1] the probabilities of  $E_1, E_2$  vanish as  $\exp(-\text{const}N^2)$ . By Cauchy Schwartz using that  $M^{(1)}$  and  $M^{(2)}$  have the same distribution and roughly estimating we see that

$$(2.6) \quad \mathbb{E} \left[ \prod_{i=1}^2 \det \left( M_{N-1}^{(i)}(\alpha) \right)^2 \right] \leq \mathbb{E} \left[ (|\lambda|_{\max}(M^{(1)}(\alpha)))^{4(N-1)} \right].$$

That this is bounded by  $\exp(\text{const}N)$  follows immediately from the tail estimate

$$(2.7) \quad \mathbb{P} \left( |\lambda|_{\max}(M^{(1)}(\alpha)) > t \right) \leq \exp(-\text{const}t^2 N).$$

This is easily obtained from the fact that  $M^{(1)}(\alpha)$  is defined as

$$\begin{pmatrix} G_{N-2} & \vec{0} \\ \vec{0}^T & 0 \end{pmatrix} + \begin{pmatrix} 0_{N-2} & Z^{(1)}(\alpha) \\ Z^{(1)}(\alpha)^T & 0 \end{pmatrix} + \begin{pmatrix} 0 & \vec{0} \\ \vec{0}^T & Q^{(i)}(\alpha) + \sqrt{\frac{N}{N-1}\xi''} m_i(\alpha, 1, 1) \end{pmatrix} - \sqrt{\frac{1}{(N-1)\xi''}} X_1(\alpha) I_{N-1},$$

and that all terms have operator norm with tails satisfying (2.7). Here the top left entry always has dimension  $(N-2) \times (N-2)$  and the bottom right  $1 \times 1$ . Also  $G_{N-2}$  a normalized GOE matrix. Lastly  $I_{N-1}$  is the dimension  $N-1$  identity matrix and  $0_{N-2}, \vec{0}$  are the zero matrix and vector respectively. For more details see [BDG01, (4.9) and (4.13)]. The first term is a normalized GOE matrix and the claimed tail estimate is given by e.g. [BDG01, Lemma

6.3]. The second term has operator norm which is up to multiplicative constant the root of a  $\chi^2(N-1)$  distributed random variable. Since the rate function of the  $\chi^2(1)$  distribution is asymptotically linear at  $\infty$  the claimed tail bound follows. The operator norms of the third and fourth terms are the absolute value of a Gaussian with variance of order  $N^{-1}$ , which have tails as claimed, which concludes the argument that (2.6) is at most  $\exp(\text{const}N)$  and therefore yields the claim.  $\square$

### 3. MAXIMIZATION OF TWO POINT ANNEALED COMPLEXITY FORMULA

In this section we prove that the two point complexity  $S$  on  $[0, 1]$  is maximized at  $\alpha = 0$ .

**Theorem 3.1** (Second moment complexity formula is maximized at  $\alpha = 0$ ). *For all  $x \in \mathbb{R}$  and all mixtures not of form  $\xi(x) = cx^2$  we have*

$$I(x) \geq 0 \Rightarrow \sup_{\alpha \in [0,1]} S(x, \alpha) = S(x, 0) = 2I(x).$$

**Remark 3.2.** a) *We will later in the proof of Theorem 1.1 use simple and general geometric considerations to show that  $\alpha < 0$  can be ignored, see (4.1) and Remark 4.1 below for details.*

b) *The fact that  $\alpha = 0$  maximizes  $S$  means that most pairs of local maxima of given radial derivative are approximately orthogonal. In the terminology of spin glasses one can say that they are “replica symmetric”.*

The proof we give here is elementary in nature but by no means trivial. It is guided by seeking a sense of algebraic beauty and could not have been reasonably derived without use of numerical checks and computer algebra. While checking step by step is elementary it is hard even for the authors to see a guiding principle.

**Proof of Theorem 3.1.** Let  $\alpha \in (0, 1)$  and  $\xi$  be a mixture not of form  $\xi(x) = cx^2$  and consider

$$(3.1) \quad S(x, 0) - S(x, \alpha) = \frac{1}{2} \ln \left( \frac{\xi'^2 - \xi_\alpha'^2}{(1 - \alpha^2)\xi'^2} \right) + \frac{x^2}{2\xi''} (Q(0) - Q(\alpha)).$$

By the definition (2.2) of  $Q$  we have

$$Q(0) - Q(\alpha) = 2\xi'' \left( \frac{\xi' - \alpha\xi'_\alpha + (1 - \alpha^2)\xi''_\alpha}{\xi'^2 - \xi_\alpha'^2 + (\xi' - \alpha\xi'_\alpha)(\xi'' + \xi''_\alpha) + (1 - \alpha^2)\xi''\xi''_\alpha} - \frac{1}{\xi' + \xi''} \right).$$

Since

$$\begin{aligned} & (\xi' + \xi'') (\xi' - \alpha\xi'_\alpha + (1 - \alpha^2)\xi''_\alpha) - \left( \xi'^2 - \xi_\alpha'^2 + (\xi' - \alpha\xi'_\alpha)(\xi'' + \xi''_\alpha) + (1 - \alpha^2)\xi''\xi''_\alpha \right) \\ &= -\alpha\xi'_\alpha\xi' - \xi'\alpha^2\xi''_\alpha + \xi_\alpha'^2 + \alpha\xi'_\alpha\xi''_\alpha \\ &= (\xi'_\alpha - \alpha\xi') (\xi'_\alpha + \alpha\xi''_\alpha) \end{aligned},$$

we obtain

$$(3.2) \quad Q(0) - Q(\alpha) = \frac{2\xi''}{\xi' + \xi''} \frac{2\xi''(\xi'_\alpha - \alpha\xi')(\xi'_\alpha + \alpha\xi''_\alpha)}{\xi'^2 - \xi_\alpha'^2 + (\xi' - \alpha\xi'_\alpha)(\xi'' + \xi''_\alpha) + (1 - \alpha^2)\xi''\xi''_\alpha} < 0,$$

where the negativity is due to  $\xi'_\alpha - \alpha\xi'$  being the only negative term. The next lemma will be used to bound the term  $\frac{x^2}{2\xi''}$  in (3.1).

**Lemma 3.3.** *For any mixture not of form  $\xi(x) = cx^2$  we have*

$$I(x) \geq 0 \Rightarrow 2 \leq \frac{x^2}{2\xi''} < \frac{\xi'' + \xi'}{\xi'' - \xi'} \ln \left( \frac{\xi''}{\xi'} \right).$$

*Proof.* The lower bound on  $\frac{x^2}{2\xi''}$  follows immediately from the definition of  $I$ , which is  $-\infty$  if  $\frac{x^2}{2\xi''} < 2$ . Assume  $\frac{x^2}{2\xi''} \geq \frac{\xi'' + \xi'}{\xi'' - \xi'} \ln\left(\frac{\xi''}{\xi'}\right)$ . We will now prove that this implies  $I(x) < 0$ . Note that  $h$  given by

$$h : [1, \infty) \rightarrow \mathbb{R} \quad x \mapsto \begin{cases} \frac{x+1}{x-1} \ln(x) & \text{for } x > 1, \\ 2 & \text{for } x = 1, \end{cases}$$

is continuous and strictly increasing. Using that the mixture is not of form  $cx^2$  we have  $\xi'' > \xi'$  and therefore

$$\frac{x^2}{2\xi''} \geq \frac{\xi'' + \xi'}{\xi'' - \xi'} \ln\left(\frac{\xi''}{\xi'}\right) = h\left(\frac{\xi''}{\xi'}\right) > h(1) = 2.$$

Using that  $\Omega'(y) = -\sqrt{y^2 - 2}$  and  $\Omega(\sqrt{2}) = 0$  we have  $\Omega(y) < 0$  for  $y > \sqrt{2}$ , which gives

$$I(x) = \frac{1}{2} \ln\left(\frac{\xi''}{\xi'}\right) - \frac{x^2}{4\xi''} \frac{\xi'' - \xi'}{\xi'' + \xi'} + \Omega\left(\frac{x}{\sqrt{2\xi''}}\right) < \frac{1}{2} \ln\left(\frac{\xi''}{\xi'}\right) - \frac{\xi'' + \xi'}{\xi'' - \xi'} \ln\left(\frac{\xi''}{\xi'}\right) \frac{1}{2} \frac{\xi'' - \xi'}{\xi'' + \xi'} = 0. \quad \square$$

Applying Lemma 3.3 to (3.1) while recalling (3.2) we have

$$(3.3) \quad S(x, 0) - S(x, \alpha) > \frac{1}{2} \ln\left(\frac{\xi'^2 - \xi_\alpha'^2}{(1 - \alpha^2)\xi'^2}\right) + \frac{\xi'' + \xi'}{\xi'' - \xi'} \ln\left(\frac{\xi''}{\xi'}\right) (Q(0) - Q(\alpha)).$$

The next lemma bounds the quantity in the log in this expression.

**Lemma 3.4.** *For  $\alpha \in [0, 1)$  we have*

$$1 \leq \frac{\xi'^2 - \xi_\alpha'^2}{(1 - \alpha^2)\xi'^2} \leq \frac{\xi''}{\xi'}.$$

*Proof.* For  $\alpha = 0$  the left inequality is sharp and for  $\alpha \rightarrow 1$  the right. Hence the proof is completed by proving that the function

$$g(\alpha) := \frac{\xi'^2 - \xi_\alpha'^2}{(1 - \alpha^2)\xi'^2}$$

is increasing. By computing  $g'$

$$g'(\alpha) = 2 \frac{\alpha(\xi'^2 - \xi_\alpha'^2) - (1 - \alpha^2)\xi'_\alpha \xi''}{(1 - \alpha^2)^2 \xi'^2}$$

we observe that  $g$  is increasing if and only if

$$(3.4) \quad \frac{\xi'^2 - \xi_\alpha'^2}{1 - \alpha^2} \geq \frac{\xi'_\alpha \xi''}{\alpha}.$$

By using that  $\xi''_\alpha$  is non negative and increasing for  $\alpha \in [0, 1]$  we obtain  $\xi' - \xi'_\alpha = \int_\alpha^1 \xi''_\alpha d\alpha \geq (1 - \alpha)\xi''_\alpha$  which implies

$$\frac{\xi'^2 - \xi_\alpha'^2}{1 - \alpha^2} \geq \xi''_\alpha \frac{\xi' + \xi'_\alpha}{1 + \alpha} = \xi''_\alpha \sum_{p \geq 2} a_p p \frac{1 + \alpha^{p-1}}{1 + \alpha}.$$

Using that  $1 \geq \alpha^{p-2}$  we further estimate

$$\geq \xi''_\alpha \sum_{p \geq 2} a_p p \frac{\alpha^{p-2} + \alpha^{p-1}}{1 + \alpha} = \xi''_\alpha \sum_{p \geq 2} a_p p \alpha^{p-2} = \xi''_\alpha \frac{\xi'_\alpha}{\alpha},$$



which yields (3.4).  $\square$

We can use Lemma 3.4 together with the fact that (the continuous continuation on  $[1, \infty)$  of)  $\frac{\sqrt{z}}{z-1} \ln(z)$  is decreasing together with to obtain

$$\frac{\sqrt{\frac{\xi'^2 - \xi_\alpha'^2}{(1-\alpha^2)\xi'^2}}}{\frac{\xi'^2 - \xi_\alpha'^2}{(1-\alpha^2)\xi'^2} - 1} \ln\left(\frac{\xi'^2 - \xi_\alpha'^2}{(1-\alpha^2)\xi'^2}\right) \geq \frac{\sqrt{\frac{\xi''}{\xi'}}}{\frac{\xi''}{\xi'} - 1} \ln\left(\frac{\xi''}{\xi'}\right).$$

Applying this to (3.3) we obtain

$$S(x, 0) - S(x, \alpha) > \ln\left(\frac{\xi''}{\xi'}\right) \left( \frac{1}{2} \frac{\sqrt{\frac{\xi''}{\xi'}} \left( \frac{\xi'^2 - \xi_\alpha'^2}{(1-\alpha^2)\xi'^2} - 1 \right)}{\sqrt{\frac{\xi'^2 - \xi_\alpha'^2}{(1-\alpha^2)\xi'^2}} \left( \frac{\xi''}{\xi'} - 1 \right)} + \frac{\xi'' + \xi'}{\xi'' - \xi'} (Q(0) - Q(\alpha)) \right).$$

Since  $\ln\left(\frac{\xi''}{\xi'}\right) > 0$  it suffices to show that

$$4\sqrt{\frac{\xi'^2 - \xi_\alpha'^2}{(1-\alpha^2)\xi'^2}} \leq \frac{\sqrt{\frac{\xi''}{\xi'}} \left( \frac{\xi'^2 - \xi_\alpha'^2}{(1-\alpha^2)\xi'^2} - 1 \right)}{\left( \frac{\xi''}{\xi'} - 1 \right) \frac{\xi'' + \xi'}{\xi'' - \xi'} \frac{1}{2} (Q(\alpha) - Q(0))}.$$

Multiplying both sides by  $(1-\alpha^2)\sqrt{\frac{\xi''}{\xi'}}$  and canceling terms this reads

$$(3.5) \quad 4\sqrt{1 - \left(\frac{\xi'_\alpha}{\xi'}\right)^2} \sqrt{1 - \alpha^2} \sqrt{\frac{\xi''}{\xi'}} \leq \frac{1}{\xi'^2} \frac{\alpha^2 \xi'^2 - \xi_\alpha'^2}{\frac{\xi'' + \xi'}{\xi'' - \xi'} \frac{1}{2} (Q(\alpha) - Q(0))}.$$

Plugging in the equality in (3.2) the right hand side reads

$$\frac{\alpha^2 - \left(\frac{\xi'_\alpha}{\xi'}\right)^2}{(\alpha\xi' - \xi'_\alpha)(\xi'_\alpha + \alpha\xi''_\alpha)} \left( \xi'^2 - \xi_\alpha'^2 + (\xi' - \alpha\xi'_\alpha)(\xi'' + \xi''_\alpha) + (1-\alpha^2)\xi''\xi''_\alpha \right).$$

By multiplying the fraction and dividing the bracket by  $\xi'^2$  and canceling we obtain equality to

$$= \frac{\alpha\xi' + \xi'_\alpha}{\xi'_\alpha + \alpha\xi''_\alpha} \left( 1 - \frac{\xi_\alpha'^2}{\xi'^2} + (1-\alpha^2) \frac{\xi''\xi''_\alpha}{\xi'^2} + (1-\alpha) \frac{\xi'_\alpha}{\xi'} \frac{\xi'' + \xi''_\alpha}{\xi'} \right).$$

Lemma 3.5 stated immediately below implies (3.5), by adding (3.6) and (3.7). Therefore the proof of Theorem 3.1 is finished once we have proved Lemma 3.5.

**Lemma 3.5.** *For  $\alpha \in (0, 1)$ , we have*

$$(3.6) \quad 2\sqrt{1 - \left(\frac{\xi'_\alpha}{\xi'}\right)^2} \sqrt{1 - \alpha^2} \sqrt{\frac{\xi''}{\xi'}} \leq \left( 1 - \alpha \frac{\xi'_\alpha}{\xi'} \right) \frac{\xi'' + \xi''_\alpha}{\xi'} \frac{\alpha\xi' + \xi'_\alpha}{\xi'_\alpha + \alpha\xi''_\alpha}$$

as well as

$$(3.7) \quad 2\sqrt{1 - \left(\frac{\xi'_\alpha}{\xi'}\right)^2} \sqrt{1 - \alpha^2} \sqrt{\frac{\xi''}{\xi'}} \leq \left( 1 - \frac{\xi_\alpha'^2}{\xi'^2} + (1-\alpha^2) \frac{\xi''\xi''_\alpha}{\xi'^2} \right) \frac{\alpha\xi' + \xi'_\alpha}{\xi'_\alpha + \alpha\xi''_\alpha}.$$

Inequalities (3.6) and (3.7) certainly look artificial at first glance, but a surprising structure is in fact hidden under the surface, which is brought to the fore by the following Lemma 3.6.

The proof of Lemma 3.5 requires certain key observations discussed by Lemmata 3.6, 3.9 and 3.8. Hence we prove these Lemmata first and only then return to the proof of Lemma 3.5.

**Lemma 3.6.** *With  $f(x) = x + \frac{1}{x}$  we have*

$$(3.6) \Leftrightarrow 2f(\sqrt{W}) \leq f(\sqrt{U})f(\sqrt{V}),$$

$$\text{for } U = \frac{\xi''}{\xi_\alpha}, V = \frac{\alpha}{\xi'_\alpha} \frac{\xi'^2 - \xi_\alpha'^2}{(1 - \alpha^2)\xi'}, W = \frac{\xi'_\alpha}{\alpha\xi''_\alpha}.$$

as well as

$$(3.7) \Leftrightarrow 2f(\sqrt{Z}) \leq f(\sqrt{X})f(\sqrt{Y}),$$

$$\text{for } X = \frac{\xi'^2 - \xi_\alpha'^2}{\xi''\xi''_\alpha(1 - \alpha^2)}, Y = \frac{\alpha\xi'}{\xi'_\alpha}, Z = \frac{\xi'_\alpha}{\alpha\xi''_\alpha}.$$

*Proof.* We start deriving the first equivalence by writing out (3.6), replacing all instances of  $\xi''$  by  $U\xi''_\alpha$ , as well as dividing by  $\sqrt{U}$ , which yields

$$2\sqrt{1 - \left(\frac{\xi'_\alpha}{\xi'}\right)^2} \sqrt{1 - \alpha^2} \sqrt{\frac{\xi''_\alpha}{\xi'}} \leq \left(1 - \alpha \frac{\xi'_\alpha}{\xi'}\right) \frac{f(\sqrt{U})\xi''_\alpha(\alpha\xi' + \xi'_\alpha)}{\xi'(\xi'_\alpha + \alpha\xi''_\alpha)}.$$

Next we replace all instances of  $\xi''_\alpha$  by  $\frac{\xi'_\alpha}{\alpha W}$ ; multiply by  $W+1$  and use  $\left(1 - \alpha \frac{\xi'_\alpha}{\xi'}\right) \frac{1}{\alpha\xi'} (\alpha\xi' + \xi'_\alpha) = 1 - \frac{\xi_\alpha'^2}{\xi'^2} + (1 - \alpha^2) \frac{\xi'_\alpha}{\alpha\xi'}$  to obtain

$$2f(\sqrt{W}) \sqrt{1 - \left(\frac{\xi'_\alpha}{\xi'}\right)^2} \sqrt{1 - \alpha^2} \sqrt{\frac{\xi'_\alpha}{\alpha\xi'}} \leq f(\sqrt{U}) \left(1 - \frac{\xi_\alpha'^2}{\xi'^2} + (1 - \alpha^2) \frac{\xi'_\alpha}{\alpha\xi'}\right).$$

Dividing by the roots on the left hand side and using the definition of  $V$  we easily obtain equivalence to

$$2f(\sqrt{W}) \leq f(\sqrt{U})f(\sqrt{V})$$

as claimed.

For the second equivalence we proceed in the same fashion. Writing down (3.7), replacing all instances of  $\xi''$  with  $\frac{\xi'^2 - \xi_\alpha'^2}{X\xi''_\alpha(1 - \alpha^2)}$  and multiplying by  $\sqrt{X}$  we have

$$2 \left(1 - \left(\frac{\xi'_\alpha}{\xi'}\right)^2\right) \sqrt{\frac{\xi'}{\xi''_\alpha}} \leq f(\sqrt{X}) \left(1 - \frac{\xi_\alpha'^2}{\xi'^2}\right) \frac{\alpha\xi' + \xi'_\alpha}{\xi'_\alpha + \alpha\xi''_\alpha}.$$

After removing the factor  $1 - \frac{\xi_\alpha'^2}{\xi'^2}$  from both sides we replace all remaining instances of  $\xi''_\alpha$  with  $\frac{\xi'_\alpha}{\alpha Z}$  and multiply by  $1 + \frac{1}{Z}$  to obtain

$$2f(Z) \sqrt{\frac{\alpha\xi'}{\xi'_\alpha}} \leq f(\sqrt{X}) \frac{\alpha\xi' + \xi'_\alpha}{\xi'_\alpha}.$$

Reorganizing the remaining terms and replacing with  $Y$  we immediately obtain the claim.  $\square$

The first part of Lemma 3.5 will follow easily from the next lemma.

**Lemma 3.7.** *For all  $\alpha \in (0, 1)$  it holds that*

$$V \geq W^{-1} \geq 1,$$

for  $V, W$  as in Lemma 3.6.

*Proof.* Since we only consider  $p \geq 2$  it is easy to check that  $W^{-1} \geq 1$ . The claim  $V \geq W^{-1}$  is equivalent to

$$(3.8) \quad \frac{\xi'_\alpha{}^2}{\xi'^2} + (1 - \alpha^2) \frac{\xi''_\alpha}{\xi'} \leq 1.$$

Consider the distribution given by  $\mathbb{P}(P = p) \propto a_p p$  and write (3.8) as follows

$$\mathbb{E}[\alpha^{P-1}]^2 + (1 - \alpha^2) \mathbb{E}[(P-1)\alpha^{P-2}] \leq 1.$$

This inequality follows by estimating  $\mathbb{E}_Q[\alpha^{P-1}]^2 \leq \mathbb{E}_Q[\alpha^{2P-2}]$  and realizing that

$$(3.9) \quad \alpha^{2P-2} + (1 - \alpha^2)(P-1)\alpha^{P-2} \leq 1$$

for all  $\alpha \in [0, 1]$  and  $P \geq 2$ .

To verify (3.9) first note for  $\alpha \in [0, 1]$

$$(3.10) \quad \frac{\partial}{\partial \alpha} (\alpha^{2P-2} + (1 - \alpha^2)(P-1)\alpha^{P-2}) = (P-1)\alpha^{P-3} (2\alpha^P - 2\alpha^2 + (P-2)(1 - \alpha^2)).$$

Letting

$$h_P(\alpha) := 2\alpha^P - 2\alpha^2 + 2\alpha^2 + (P-2)(1 - \alpha^2) = 2\alpha^P + P - 2 - P\alpha^2,$$

we have

$$h'_P(\alpha) = 2P(\alpha^{P-1} - \alpha) \leq 0,$$

and therefore  $h_P(\alpha) \geq h_P(1) = 0$ . From this we see that (3.10) is non-negative, so (3.8) follows since it trivially holds for  $\alpha = 1$ .  $\square$

We are now ready to prove the first part of Lemma 3.5.

**Proof of (3.6).** By Lemma 3.6 and using its notation it suffices to show that

$$2f(\sqrt{W}) \leq f(\sqrt{U})f(\sqrt{V}).$$

Clearly  $f$  is always at least 2 and therefore this in turns follows from

$$f(\sqrt{W}) \leq f(\sqrt{V}),$$

which is a consequence of Lemma 3.7. This completes the proof of (3.6).  $\square$

Before proving the second part of Lemma 3.5 we prove two inequalities involving

$$S = \frac{Y+1}{2} \left( 1 - \frac{\alpha Y - 1}{2Y} \right).$$

The first is:

**Lemma 3.8.** *We have for  $\alpha \in (0, 1)$*

$$(3.11) \quad 2f(\sqrt{Z}) \leq f(\sqrt{SZ})f(\sqrt{Y}).$$

*Proof.* Dividing (3.11) by  $\sqrt{Z}$  we obtain the equivalent representation

$$2 + 2Z^{-1} \leq \sqrt{S}f(\sqrt{Y}) + Z^{-1} \frac{f(\sqrt{Y})}{\sqrt{S}}.$$

Hence it suffices to show

$$Z^{-1} \left( 2 - \frac{f(\sqrt{Y})}{\sqrt{S}} \right) \leq \sqrt{S}f(\sqrt{Y}) - 2.$$

By observing that

$$(3.12) \quad \frac{f(\sqrt{Y})^2}{4S} = \frac{Y + 2 + Y^{-1}}{2(Y + 1) \left(1 - \frac{\alpha}{2} \frac{Y-1}{Y}\right)} = 1 - \frac{(1 - \alpha)(Y - Y^{-1})}{2(Y + 1) \left(1 - \frac{\alpha}{2} \frac{Y-1}{Y}\right)} < 1,$$

since  $\alpha \in (0, 1)$  and  $Y > 1$ , we obtain  $2 - \frac{f(\sqrt{Y})}{\sqrt{S}} \geq 0$ . Therefore (3.11) is also equivalent to

$$(3.13) \quad Z^{-1} \leq \frac{\sqrt{S}f(\sqrt{Y}) - 2}{2 - \frac{f(\sqrt{Y})}{\sqrt{S}}}.$$

Reformulating the right hand side of this inequality yields

$$(3.14) \quad \begin{aligned} \frac{\sqrt{S}f(\sqrt{Y}) - 2}{2 - \frac{f(\sqrt{Y})}{\sqrt{S}}} &= \frac{(\sqrt{S}f(\sqrt{Y}) - 2)(2 + \frac{f(\sqrt{Y})}{\sqrt{S}})}{4 - \frac{f(\sqrt{Y})^2}{S}} = \frac{S(f(\sqrt{Y})^2 - 4) + 2f(\sqrt{Y})(S - 1)\sqrt{S}}{4S - f(\sqrt{Y})^2} \\ &= \frac{\left\{(S - 1)f(\sqrt{Y})^2 + f(\sqrt{Y})^2 - 4S\right\} + (S - 1)f(\sqrt{Y})^2 \sqrt{\frac{4S}{f(\sqrt{Y})^2}}}{4S - f(\sqrt{Y})^2}. \end{aligned}$$

We then use the easily checked representations

$$S = \frac{(Y + 1)^2}{4Y} \left(1 + (1 - \alpha) \frac{Y - 1}{Y + 1}\right) \text{ and } f(\sqrt{Y})^2 = \frac{(Y + 1)^2}{Y}$$

to compute (3.14) piece by piece as follows

$$\begin{aligned} S - 1 &= \frac{Y - 1}{4Y} ((2 - \alpha)Y - \alpha), \\ (S - 1)f(\sqrt{Y})^2 &= (Y(2 - \alpha) - \alpha) \frac{(Y - 1)(Y + 1)^2}{4Y^2}, \\ \frac{4S}{f(\sqrt{Y})^2} &= 1 + (1 - \alpha) \frac{Y - 1}{Y + 1}, \\ 4S - f(\sqrt{Y})^2 &= (1 - \alpha) \frac{(Y - 1)(Y + 1)}{Y}. \end{aligned}$$

Using the computations so far we also obtain for the curly bracket in (3.14)

$$(S - 1)f(\sqrt{Y})^2 - 4S + f(\sqrt{Y})^2 = ((Y(2 - \alpha) - \alpha)(Y + 1) - 4(1 - \alpha)Y) \frac{(Y - 1)(Y + 1)}{4Y^2},$$

where the first factor on the LHS equals

$$\begin{aligned} &(Y(1 + 1 - \alpha) - 1 + 1 - \alpha)(Y + 1) - 4(1 - \alpha)Y \\ &= (Y - 1)(Y + 1) + (1 - \alpha)\{(Y + 1)\{Y + 1\} - 4Y\} \\ &= Y^2 - 1 + (1 - \alpha)(Y - 1)^2 = (Y^2 - 1) \left(1 + (1 - \alpha) \frac{Y-1}{Y+1}\right). \end{aligned}$$

Collecting the pieces of (3.14) we computed and multiplying numerator and denominator by  $\frac{4Y^2}{(Y+1)(Y-1)}$  we obtain

$$\frac{\sqrt{S}f(\sqrt{Y}) - 2}{2 - \frac{f(\sqrt{Y})}{\sqrt{S}}} = \frac{(Y^2 - 1) \left(1 + (1 - \alpha) \frac{Y-1}{Y+1}\right) + (Y(2 - \alpha) - \alpha)(Y + 1) \sqrt{1 + (1 - \alpha) \frac{Y-1}{Y+1}}}{4(1 - \alpha)Y}.$$

By using the trivial estimate  $1 + (1 - \alpha)\frac{Y-1}{Y+1} \geq 1$  (recall  $Y > 1$ ) twice we clearly have

$$\frac{\sqrt{S}f(\sqrt{Y}) - 2}{2 - \frac{f(\sqrt{Y})}{\sqrt{S}}} \geq \frac{Y^2 - 1 + (Y(2 - \alpha) - \alpha)(Y + 1)}{4(1 - \alpha)Y} = \frac{1}{4} \frac{Y + 1}{Y} \left( 2 \frac{Y - 1}{1 - \alpha} + Y + 1 \right).$$

Estimating further using  $Y + 1 \geq 2$  then yields

$$(3.15) \quad \frac{\sqrt{S}f(\sqrt{Y}) - 2}{2 - \frac{f(\sqrt{Y})}{\sqrt{S}}} \geq \frac{1}{2} \frac{Y + 1}{Y} \left( \frac{Y - 1}{1 - \alpha} + 1 \right).$$

It remains to show that

$$(3.16) \quad Z^{-1} \leq \frac{1}{2} \frac{Y + 1}{Y} \left( \frac{Y - 1}{1 - \alpha} + 1 \right),$$

since then (3.13) and thereby the claim immediately follow from (3.15). By definition of  $Y$  and  $Z$ , after dividing both sides by  $Y$ , (3.16) reads

$$(3.17) \quad \frac{\alpha \xi''_{\alpha}}{\xi'} \leq \frac{1}{2} \frac{\xi'_{\alpha}}{\alpha \xi'} \left( 1 + \frac{\xi'_{\alpha}}{\alpha \xi'} \right) \left( \frac{\frac{\alpha \xi'_{\alpha}}{\xi'} - 1}{1 - \alpha} + 1 \right) = \frac{1}{2} \left( 1 + \frac{\xi'_{\alpha}}{\alpha \xi'} \right) \frac{1 - \frac{\xi'_{\alpha}}{\xi'}}{1 - \alpha}.$$

With  $P \geq 2$  a random variable with  $\mathbb{P}(P = p) \propto a_p p$  we can write (3.17) as

$$(3.18) \quad \mathbb{E}[(P - 1)\alpha^{P-2}] \leq \frac{1}{2} (1 + \mathbb{E}[\alpha^{P-2}]) \frac{1 - \mathbb{E}[\alpha^{P-1}]}{1 - \alpha}.$$

Multiplying by  $2(1 - \alpha)$  and bringing all terms except the 1 to the left hand side we have the equivalent representation

$$\mathbb{E}[2(P - 1)(1 - \alpha)\alpha^{P-2}] - (1 - \alpha)\mathbb{E}[\alpha^{P-2}] + \alpha\mathbb{E}[\alpha^{P-2}]^2 \leq 1.$$

But by Cauchy Schwartz inequality and linearity of expectation

$$\mathbb{E}[2(P - 1)(1 - \alpha)\alpha^{P-2}] - (1 - \alpha)\mathbb{E}[\alpha^{P-2}] + \alpha\mathbb{E}[\alpha^{P-2}]^2 \leq \mathbb{E}[\alpha^{P-2} ((2P - 3)(1 - \alpha) + \alpha^{P-1})].$$

Then (3.18) and (3.17) follows once we have show that

$$(3.19) \quad \alpha^{P-2} ((2P - 3)(1 - \alpha) + \alpha^{P-1}) \leq 1 \text{ for all } \alpha \in [0, 1], P \geq 2.$$

To this end note that

$$\frac{\partial}{\partial \alpha} (\alpha^{P-2} (\alpha^{P-1} + (2P - 3)(1 - \alpha))) = (2P - 3)\alpha^{P-3} h_P(\alpha)$$

for

$$h_P(\alpha) := P - 2 - (P - 1)\alpha + \alpha^{P-1},$$

and  $h_P(\alpha) \geq h_P(1) = 0$  by checking that  $h'_P(\alpha) \leq 0$ . Since (3.19) trivially holds for  $\alpha = 1$  this proves (3.19), and finishes the proof.  $\square$

The second inequality we need for the second second part of Lemma 3.5 is:

**Lemma 3.9.** *We have for  $\alpha \in (0, 1)$*

$$\xi''_{\alpha} \geq \xi' \Rightarrow f(\sqrt{SZ}) \leq f(\sqrt{X}).$$

*Proof.* The claim follows immediately from

$$X \leq SZ \leq 1.$$

The second inequality is quickly checked by writing it out partially

$$SZ = \frac{\alpha \xi' + \xi'_\alpha}{2\alpha \xi''_\alpha} \left( 1 - \frac{\alpha Y - 1}{2Y} \right)$$

and observing that  $\xi' \leq \xi''_\alpha$  by assumption,  $\xi'_\alpha \leq \alpha \xi''_\alpha$  trivially, remembering that  $Y \geq 1$  and checking that both factors are at most 1.

It remains to check  $X \leq SZ$ . By definition

$$X = \frac{\xi'^2 - \xi_\alpha'^2}{\xi'' \xi''_\alpha (1 - \alpha^2)} = \frac{\int_\alpha^1 \xi''(t) dt}{1 - \alpha} \frac{\xi' + \xi'_\alpha}{\xi'' \xi''_\alpha (1 + \alpha)}.$$

Since  $\alpha \in [0, 1]$  and  $\xi''$  is convex on  $[0, 1]$  we have  $\frac{\int_\alpha^1 \xi''(t) dt}{1 - \alpha} \leq \frac{\xi'' + \xi''_\alpha}{2}$  and so

$$X \leq \frac{\xi'' + \xi''_\alpha}{2} \frac{\xi' + \xi'_\alpha}{\xi'' \xi''_\alpha (1 + \alpha)} = \frac{\xi'_\alpha}{2\alpha \xi''_\alpha} \left( 1 + \frac{\xi''_\alpha}{\xi''} \right) \left( 1 + \frac{\xi'}{\xi'_\alpha} \right) \frac{\alpha}{1 + \alpha}.$$

Consider random variables  $P, Q \geq 2$  with  $\mathbb{P}(P = p) \propto a_p p$  and  $\mathbb{P}(Q = p) \propto a_p p(p - 1)$ . Clearly  $\mathbb{P}(P \geq k) \leq \mathbb{P}(Q \geq k)$  and so since  $\alpha \in (0, 1)$  and  $P \geq 2$

$$(3.20) \quad \frac{\xi''_\alpha}{\xi''} = \mathbb{E}[\alpha^{Q-2}] \leq \mathbb{E}[\alpha^{P-2}] = \frac{\xi'_\alpha}{\alpha \xi'}.$$

Thus

$$X \leq \frac{\xi'_\alpha}{2\alpha \xi''_\alpha} \left( 1 + \frac{\xi'_\alpha}{\alpha \xi'} \right) \left( 1 + \frac{\xi'}{\xi'_\alpha} \right) \frac{\alpha}{1 - \alpha} = \frac{Z}{2} (1 + Y^{-1}) \frac{\alpha + Y}{1 + \alpha} = Z \frac{Y + 1}{2} \left( 1 - \frac{\alpha}{1 + \alpha} \frac{Y - 1}{Y} \right).$$

Remembering that  $Y \geq 1$  and estimating  $1 + \alpha \leq 2$  yields  $X \leq ZS$ , i.e. the claim.  $\square$

We are now ready to prove the second part of Lemma 3.5.

**Proof of (3.7).** By Lemma 3.6 we need to prove

$$2f(\sqrt{Z}) \leq f(\sqrt{X})f(\sqrt{Y}).$$

To this end first note that if  $\xi' \geq \xi''_\alpha$  the claim follows immediately, since then  $Y \geq Z^{-1} \geq 1$  and therefore  $f(\sqrt{Y}) \geq f(\sqrt{Z})$ , using that  $f(x) \geq 2$ . Hence we can assume  $\xi' < \xi''_\alpha$  and by Lemma 3.8 and Lemma 3.9 the claim follows.  $\square$

This completes the proof of Lemma 3.5, and as explained above the statement of that lemma, it also completes the proof of Theorem 3.1.  $\square$

#### 4. PROOF OF THE MAIN RESULTS

This section is devoted to the proof of Theorem 1.1.

**Proof of Theorem 1.1.** Equation (1.5) follows immediately from Lemma 2.1 and Markov inequality, recalling (1.3). When  $\xi(x) = cx^2$  one can check that  $S(x, \alpha) = 2\Omega(\frac{x}{2\sqrt{c}})$  for all  $\alpha$  so the claim trivially follows. Hence we assume the contrary for the remainder of the proof.

To verify (1.4) note that for any measurable set  $D$  by Cauchy Schwartz inequality we have

$$\mathbb{E}[\mathcal{N}(D)]^2 \leq \mathbb{E}[\mathcal{N}(D)^2],$$

hence it is sufficient to show that the reverse inequality holds on exponential scale.

To this end note that for any  $\delta > 0$

$$(4.1) \quad 0 \leq \left| \sum_{\sigma \in \mathcal{N}(D)} \sigma \right|_2^2 = \sum_{(\sigma, \tau) \in \mathcal{N}_2(D, [-1, 1])} \sigma \tau \leq \mathcal{N}_2(D, (-\delta, 1]) - \delta \mathcal{N}_2(D, [-1, -\delta])$$

and therefore

$$(4.2) \quad \mathcal{N}_2(D, [-1, 1]) \leq \left(\frac{1}{\delta} + 1\right) \mathcal{N}_2(D, [-\delta, 1]).$$

Using  $\mathcal{N}_2(D, [-1, 1]) = \mathcal{N}(D)^2$  we obtain

$$\begin{aligned} & \lim_{\varepsilon \searrow 0} \lim_{N \rightarrow \infty} \frac{1}{N} \ln \mathbb{E}[\mathcal{N}((x - \varepsilon, x + \varepsilon))^2] \\ & \leq \lim_{\varepsilon \searrow 0} \lim_{N \rightarrow \infty} \frac{1}{N} \ln \mathbb{E}[\mathcal{N}((x - \varepsilon, x + \varepsilon)) + \mathcal{N}_2((x - \varepsilon, x + \varepsilon), [-\delta, 1])]. \end{aligned}$$

Applying Lemmata 2.1 and 2.2 yields that this is at most

$$\max \left\{ \sup_{\alpha \in [-\delta, 1]} S(x, \alpha), I(x) \right\}.$$

Since  $S$  is continuous in  $\alpha$  we may optimize over  $\delta > 0$ . Furthermore by assumption  $x \in [r_\infty, r_0]$  and therefore  $S(x, 0) = 2I(x) \geq I(x) \geq 0$ . Hence we obtain overall

$$\lim_{\varepsilon \searrow 0} \lim_{N \rightarrow \infty} \frac{1}{N} \ln (\mathbb{E}[\mathcal{N}((x - \varepsilon, x + \varepsilon))^2]) \leq \sup_{\alpha \in [0, 1]} S(x, \alpha).$$

By Theorem 3.1 this supremum is attained in  $\alpha = 0$  and the claim follows since

$$\sup_{\alpha \in [0, 1]} S(x, \alpha) = S(x, 0) = 2I(x) = \lim_{\varepsilon \searrow 0} \lim_{N \rightarrow \infty} \frac{2}{N} \ln (\mathbb{E}[\mathcal{N}((x - \varepsilon, x + \varepsilon))]).$$

□

**Remark 4.1.** *The estimate (4.2) can be improved to*

$$\mathbb{E}[\mathcal{N}_2(D, [-1, 1])] \leq (1 + o(1)) \mathbb{E}[\mathcal{N}_2(D, [-\delta, 1])]$$

by replacing (4.1) with

$$0 \leq \left| \sum_{\sigma \in \mathcal{N}(D)} \sigma \right|_2^2 = \sum_{(\sigma, \tau) \in \mathcal{N}_2(D, [-1, 1])} \sigma \tau \leq \delta^2 \mathcal{N}_2(D, (-\delta, \delta^2]) - \delta \mathcal{N}_2(D, [-1, -\delta]) + \mathcal{N}_2(D, (\delta^2, 1])$$

and applying Lemmata 2.1, 2.2 and Theorem 3.1. While this is not necessary on an exponential scale it will be necessary if one wants to show matching of moments up to multiplicative error  $1 + o(1)$  to prove concentration on the mean as in [Sub17].

## REFERENCES

- [AB13] A. Auffinger and G. Ben Arous. “Complexity of random smooth functions on the high-dimensional sphere”. In: *Ann. Probab.* 41.6 (2013), pp. 4214–4247.
- [ABCĚ13] A. Auffinger, G. Ben Arous, and J. Černý. “Random matrices and complexity of spin glasses”. In: *Comm. Pure Appl. Math.* 66.2 (2013), pp. 165–201.
- [AT07] R. J. Adler and J. E. Taylor. *Random fields and geometry*. Springer Monographs in Mathematics. Springer, New York, 2007, pp. xviii+448. ISBN: 978-0-387-48112-8.
- [BDG01] G. Ben Arous, A. Dembo, and A. Guionnet. “Aging of spherical spin glasses”. In: *Probab. Theory Related Fields* 120.1 (2001), pp. 1–67.

- [Bel+22] D. Belius, J. Černý, S. Nakajima, and M. A. Schmidt. “Triviality of the Geometry of Mixed  $p$ -Spin Spherical Hamiltonians with External Field”. en. In: *Journal of Statistical Physics* 186.1 (Jan. 2022). Citation Key: beliusTrivialityMixedPspin2104, p. 12.
- [Bel22] D. Belius. “High temperature TAP upper bound for the free energy of mean field spin glasses”. In: *arXiv preprint arXiv:2204.00681* (2022).
- [BSZ20] G. Ben Arous, E. Subag, and O. Zeitouni. “Geometry and temperature chaos in mixed spherical spin glasses at low temperature: the perturbative regime”. In: *Comm. Pure Appl. Math.* 73.8 (2020), pp. 1732–1828.
- [CL04] A. Crisanti and L. Leuzzi. “Spherical  $2+p$  spin-glass model: An exactly solvable model for glass to spin-glass transition”. In: *Physical Review Letters* 93.21 (Nov. 2004). Number of pages: 4 Publisher: American Physical Society, p. 217203.
- [CS92] A. Crisanti and H. -J. Sommers. “The spherical  $p$ -spin interaction spin glass model: the statics”. en. In: *Zeitschrift für Physik B Condensed Matter* 87.3 (Oct. 1992), pp. 341–354.
- [Der80] B. Derrida. “Random-Energy Model: Limit of a Family of Disordered Models”. en. In: *Physical Review Letters* 45.2 (July 1980), pp. 79–82.
- [FN12] Y. V. Fyodorov and C. Nadal. “Critical Behavior of the Number of Minima of a Random Landscape at the Glass Transition Point and the Tracy-Widom Distribution”. en. In: *Physical Review Letters* 109.16 (Oct. 2012), p. 167203.
- [Fyo04] Y. V. Fyodorov. “Complexity of random energy landscapes, glass transition, and absolute value of the spectral determinant of random matrices”. In: *Physical review letters* 92.24 (2004), p. 240601.
- [Fyo15] Y. V. Fyodorov. “High-dimensional random fields and random matrix theory”. In: *Markov Process. Related Fields* 21.3, part 1 (2015), 483–518: Equation numbers are those of the arxiv version arxiv:1307.2379.
- [GM84] D. Gross and M. Mezard. “The simplest spin glass”. en. In: *Nuclear Physics B* 240.4 (Nov. 1984), pp. 431–452.
- [KTJ76] J. M. Kosterlitz, D. J. Thouless, and R. C. Jones. “Spherical Model of a Spin-Glass”. en. In: *Physical Review Letters* 36.20 (May 1976), pp. 1217–1220.
- [MPV87] M. Mézard, G. Parisi, and M. Virasoro. *Spin glass theory and beyond: An introduction to the replica method and its applications*. Vol. 9. World Scientific Publishing Company, 1987.
- [Pan13] D. Panchenko. *The Sherrington-Kirkpatrick model*. Springer Science & Business Media, 2013.
- [Sch42] I. J. Schoenberg. “Positive definite functions on spheres”. In: *Duke Mathematical Journal* 9.1 (Mar. 1942).
- [SK75] D. Sherrington and S. Kirkpatrick. “Solvable model of a spin-glass”. In: *Physical review letters* 35.26 (1975). Publisher: APS, p. 1792.
- [Sub17] E. Subag. “The complexity of spherical  $p$ -spin models—a second moment approach”. In: *Ann. Probab.* 45.5 (2017), pp. 3385–3450.
- [Tal00] M. Talagrand. “Multiple levels of symmetry breaking”. en. In: *Probability Theory and Related Fields* 117.4 (Aug. 2000), pp. 449–466.
- [Tal06] M. Talagrand. “Free energy of the spherical mean field model”. en. In: *Probability Theory and Related Fields* 134.3 (Mar. 2006), pp. 339–382.
- [Tal10] M. Talagrand. *Mean field models for spin glasses: Volume I: Basic examples*. Vol. 54. Springer Science & Business Media, 2010.